Improved Compressions of Cube-Connected Cycles Networks *

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Abstract

We present a new technique for the embedding of large cube-connected cycles networks (CCC) into smaller ones, a problem that arises when algorithms designed for an architecture of an ideal size are to be executed on an existing architecture of a fixed size. Using the new embedding strategy, we show that the CCC of dimension l can be embedded into the CCC of dimension k with *dilation* 1 and optimum *load* for any $k, l \in I\!N$, $k \geq 8$, such that $\frac{5}{3} + c_k < \frac{l}{k} \leq 2$, $c_k = \frac{4k+3}{3 \cdot 2^{2/3k}}$, thus improving known results. Our embedding technique also leads to improved dilation 1 embeddings in the case $\frac{3}{2} < \frac{l}{k} \leq \frac{5}{3} + c_k$.

Index Terms

Parallel computations, parallel architectures, interconnection networks, graph embedding, network simulation, cube-connected cycles network.

1 Introduction

Over the past few years, a lot of research has been done in the field of interconnection networks for parallel computer architectures (for an overview, cf. [19]). Much of the work has been focused on the capability of certain networks to simulate other network or algorithm structures, in order to execute parallel algorithms of a special structure efficiently on different processor networks (see e.g. [5, 17, 25]). One problem that is of specific interest in this context is that many existing algorithms are designed for arbitrarily large networks (see e.g. [19]), whereas, in practice, the processor network will be fixed and of smaller size. Thus, the larger network must be simulated in an efficient way on the smaller target network. There is an enormous literature on this problem (see e.g. [3, 8, 14, 15, 21, 23, 24, 26, 30]).

Customarily, the *simulation* problem is formalized as the *emdedding* problem of one graph in another (for a formal definition of the *embedding* problem, see Section 2). The "quality" of an embedding is measured by the parameters *load*, *dilation*, and *congestion*. The importance of the different parameters becomes apparent through the following result.

Proposition 1 [20]:

> *If there is an embedding of G into H with load ℓ, dilation d, and congestion c, then there is a simulation of G by H with slowdown $O(\ell + d + c)$.*

As a consequence, the load ℓ, dilation d, and congestion c have been investigated for embeddings between many common network structures like hypercubes, binary trees, meshes, shuffle-exchange networks, deBruijn networks, cube-connected cycles, butterfly networks, etc. Most of the work was done on one-to-one embeddings (for an overview, see e.g. [25, 29]), but results on many-to-one embeddings can also be found (see e.g. [2, 6, 7, 9, 12, 13, 16, 18, 22, 26, 27]). In this paper, we focus on many-to-one embeddings of the cube-connected cycles network (CCC). The CCC was introduced as a network for parallel processing in [28]. It has fixed degree, small diameter, and good routing capabilities [19]. It can execute the important class of *normal* hypercube algorithms very efficiently (see e.g. [19]). In addition, there is also a strong structural relationship to the deBruijn, shuffle-exchange, and butterfly networks [1, 10]. Hence, the efficient implementation of algorithms on CCC networks (of fixed size) is of importance. According to Proposition 1, one way of executing algorithms designed for a CCC network of arbitrary size efficiently on a CCC network of realistic (fixed) size, is to find embeddings of large CCC's into small CCC's minimizing the parameters *load*, *dilation*, and *congestion*. In this paper, we focus on *load* and *dilation*. Using our embedding strategy, many important algorithms for large CCC's can be implemented very efficiently on a CCC network of realistic size.

Many-to-one embeddings of the CCC network have been investigated in [2, 6, 12, 16, 27]. In [6, 12, 27], embeddings with optimum dilation and load are presented in the case of embedding CCC's of dimension l into k where $k|l$. The authors also restrict themselves to special kinds of embeddings of a very regular structure, like coverings [6], homogeneous emulations [12], and homomorphisms [27]. Because of the very restricted nature, Bodlaender [6] and Peine [27] are also able to classify their embeddings completely. In [2], a general procedure is described for mapping parallel algorithms into parallel architectures. This procedure is applied to the CCC network achieving dilation 1, but very high load.

Also, only special kinds of embeddings, so-called contractions, are considered. In [16], the embedding problem for CCC's is investigated taking into account general embedding functions and any possible network dimension. More precisely, it is proved that the *cube-connected cycles* network of dimension l, $CCC(l)$, can be embedded into $CCC(k)$, $l > k$, with

(a) *dilation* 2 and optimum *load* $\left\lceil \frac{l}{k} \cdot 2^{l-k} \right\rceil$.

(b) *dilation* 1 and *load*

$$\begin{cases} \left\lceil \frac{l}{k} \cdot 2^{l-k} \right\rceil & \text{for} \quad \frac{l}{k} \geq 2, \\ \left\lceil \frac{2p-1}{p} \cdot 2^{l-k} \right\rceil & \text{for} \quad p \in \{2, 3, \ldots\} \text{ such that } \frac{2p-3}{p-1} < \frac{l}{k} \leq \frac{2p-1}{p}. \end{cases}$$

In this paper, we present a new technique for the embedding of large cube-connected cycles networks into smaller ones. Using the new embedding strategy, we show:

Let $k, l \in I\!N$, $k \geq 8$, such that $\frac{5}{3} + c_k < \frac{l}{k} \leq 2$, $c_k = \frac{4k+3}{3 \cdot 2^{2/3k}}$. Then, there is a dilation 1 embedding of $CCC(l)$ into $CCC(k)$ with load $\left\lceil \frac{l}{k} \cdot 2^{l-k} \right\rceil$.

This is optimal, and improves the results from [16]. Our embedding technique also leads to improved dilation 1 embeddings in the case $\frac{3}{2} < \frac{l}{k} \leq \frac{5}{3} + c_k$.
The general strategy of the embeddings is the same as in [16], namely to map 2^{l-k} cycles in $CCC(l)$ of length l onto one cycle in $CCC(k)$ of length k and to allocate the nodes of the guest cycles as balancedly as possible on the host cycle. But in order to improve the results from [16], a completely different way of allocating the guest nodes on the host cycle is introduced.

The paper is organized as follows. Section 2 contains the definitions of the terms used in the paper. Section 3 presents the new embedding strategy. Section 4 presents the derived results. The Conclusion gives an outlook on further consequences of the new embedding technique.

2 Definitions

(Most of the terminology is taken from [19, 25].) For any graph $G = (V, E)$, let $V(G) = V$ denote the set of vertices of G, and $E(G) = E$ denote the set of edges of G. Let $\bar{a}$ denote the binary complement of $a \in \{0, 1\}$. For $\alpha = a_0 a_1 ... a_{m-1} \in \{0, 1\}^m$, let $\alpha(i) = a_0 \ldots a_{i-1} \bar{a}_i a_{i+1} \ldots a_{m-1}$.

Cube-Connected Cycles Network. The *(wrapped) cube-connected cycles network of dimension m*, denoted by $CCC(m)$, has vertex-set $V_m = \{0, 1, ..., m-1\} \times \{0, 1\}^m$, where $\{0, 1\}^m$ denotes the set of length-m binary strings. For each vertex $v = (i, \alpha) \in V_m$,

$i \in \{0, 1, ..., m-1\}, \alpha \in \{0,1\}^m$, we call i the *level* and α the *position-within-level (PWL) string* of v. The edges of $CCC(m)$ are of two types: For each $i \in \{0, 1, ..., m-1\}$ and each $\alpha = a_0 a_1 ... a_{m-1} \in \{0,1\}^m$, the vertex (i, α) on level i of $CCC(m)$ is connected

- by a *cycle-edge* with vertex $((i+1) \bmod m, \alpha)$ on level $(i+1) \bmod m$ and
- by a *cross-edge* with vertex $(i, \alpha(i))$ on level i.

For each $\alpha \in \{0,1\}^m$, the cycle

$$(0, \alpha) \leftrightarrow (1, \alpha) \leftrightarrow \ldots \leftrightarrow (m-1, \alpha) \leftrightarrow (0, \alpha)$$

of length m will be denoted by $C_\alpha(m)$ or C_α.
$CCC(m)$ has $m2^m$ nodes, $3m2^{m-1}$ edges and degree 3. An illustration of $CCC(3)$ is shown in Figure 1.

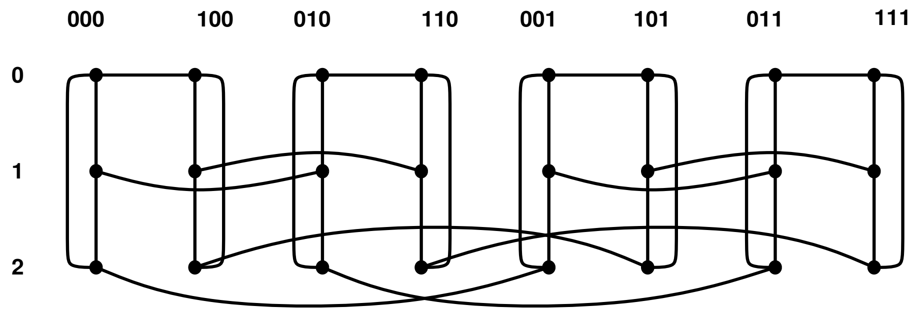

Figure 1: The cube-connected cycles $CCC(3)$

Graph Embeddings. Let G and H be finite undirected graphs. An *embedding* of G into H is a mapping f from the nodes of G to the nodes of H. G is called the *guest* graph and H is called the *host* graph of the embedding f. The *load* of the embedding f is the maximum number of vertices of the guest graph G that are mapped to the same host graph vertex. [The *optimum load* achievable is the ratio $\lceil |V(G)|/|V(H)| \rceil$ of the number of nodes in G and H.] The *dilation* of the embedding f is the maximum distance in the host between the images of adjacent guest nodes. A *routing* is a mapping r of G's edges to paths in H, $r(v_1, v_2) =$ a path from $f(v_1)$ to $f(v_2)$ in H. The *congestion* of the embedding f is the maximum number of edges that are routed through a single edge of H.

Lexicographic Orderings. Let Lex : $\{0, \ldots, m-1\} \times \{0,1\}^n \to \mathbb{N}_0$, $\text{Lex}(i, a_0 \ldots a_{n-1}) = i2^n + a_0 2^{n-1} + a_1 2^{n-2} + \ldots + a_{n-1} 2^0$. Then, the *lexicographic order* on $\{0, 1, \ldots, m-1\} \times \{0,1\}^n$ is defined by

$$(i, \alpha) < (j, \beta) \;\Leftrightarrow\; \text{Lex}(i, \alpha) < \text{Lex}(j, \beta) \ ,$$

and the *lexicographic distance* between (i, α) and (j, β) is defined as $|\text{Lex}(i, \alpha) - \text{Lex}(j, \beta)|$.

Balanced Allocations. Let $a_1, b_1, a_2, b_2 \in \mathbb{N}_0$ such that $b_1 \geq a_1$, $b_2 \geq a_2$, $b_1 - a_1 \geq b_2 - a_2$. Let $r \in \mathbb{N}$. A function

$$d : \{a_1, a_1 + 1, \ldots, b_1\} \times \{0,1\}^r \to \{a_2, a_2 + 1, \ldots, b_2\}$$

is called a *balanced allocation of* $\{a_1, \ldots, b_1\} \times \{0,1\}^r$ *among* $\{a_2, \ldots, b_2\}$ *according to the lexicographic order on* $\{a_1, \ldots, b_1\} \times \{0,1\}^r$ if d satisfies the following properties:

- $d(a_1, 0^r) = a_2, \quad d(b_1, 1^r) = b_2,$
- d is monotonic nondecreasing in the lexicographic ordering of the arguments [i.e., $d(i,\beta) \le d(i',\beta')$, if $(i,\beta) \le (i',\beta')$ according to the lexicographic order on $\{a_1, \ldots, b_1\} \times \{0,1\}^r$],
- $\left\lceil \frac{b_1 - a_1 + 1}{b_2 - a_2 + 1} \cdot 2^r \right\rceil - 1 \;\le\; |d^{-1}(j)| \;\le\; \left\lceil \frac{b_1 - a_1 + 1}{b_2 - a_2 + 1} \cdot 2^r \right\rceil$ for all $j \in \{a_2, \ldots, b_2\}$.

[Note that such an allocation function d can always be constructed for the parameters a_1, b_1, a_2, b_2, r as above.]

3 The General Embedding Strategy

The basic idea of the embeddings presented here is to map 2^{l-k} cycles $C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_{2^{l-k}}}$ in $CCC(l)$ of length l onto one cycle C_β of length k in $CCC(k)$ and to allocate the $l \cdot 2^{l-k}$ nodes of $C_{\alpha_1}, \ldots, C_{\alpha_{2^{l-k}}}$ appropriately among the k nodes of C_β.

FORMAL CONSTRUCTION:

Consider numbers $\pi(0), \pi(1), \ldots, \pi(k-1)$, where each $\pi(i) \in \{0,1,\ldots,l-1\}$, and each $\pi(i) < \pi(i+1)$. Let $\bar\pi(0), \bar\pi(1), \ldots, \bar\pi(l-k-1) \in \{0,1,\ldots,l-1\} \setminus \{\pi(0), \pi(1), \ldots, \pi(k-1)\}$ such that $\bar\pi(0) < \bar\pi(1) < \ldots < \bar\pi(l-k-1)$. [Note that $\{\pi(0), \pi(1), \ldots, \pi(k-1)\} \,\dot\cup\, \{\bar\pi(0), \bar\pi(1), \ldots, \bar\pi(l-k-1)\} = \{0,1,\ldots,l-1\}$.]

Let $a_{\pi(0)}, a_{\pi(1)}, \ldots, a_{\pi(k-1)} \in \{0,1\}$. The cycles $\{C_{a_0 a_1 \ldots a_{l-1}} \mid a_{\bar\pi(0)}, a_{\bar\pi(1)}, \ldots, a_{\bar\pi(l-k-1)} \in \{0,1\}\}$ of $CCC(l)$ are mapped onto the cycle $C_{a_{\pi(0)} a_{\pi(1)} \ldots a_{\pi(k-1)}}$ in $CCC(k)$ such that the nodes $0, 1, \ldots, l-1$ of each $C_{a_0 a_1 \ldots a_{l-1}}$ are allocated appropriately among the nodes of $C_{a_{\pi(0)} a_{\pi(1)} \ldots a_{\pi(k-1)}}$.

The exact allocation of the nodes of $\{C_{a_0 a_1 \ldots a_{l-1}} \mid a_{\bar\pi(0)}, a_{\bar\pi(1)}, \ldots, a_{\bar\pi(l-k-1)} \in \{0,1\}\}$ on $C_{a_{\pi(0)} a_{\pi(1)} \ldots a_{\pi(k-1)}}$ is determined by an allocation function

$$d : \{0,1,\ldots,l-1\} \times \{0,1\}^{l-k} \to \{0,1,\ldots,k-1\}$$

which specifies, for each node number $\in \{0,1,\ldots,l-1\}$ on the guest cycle $C_{a_0 a_1 \ldots a_{l-1}}$ and each cycle index $a_{\bar\pi(0)} a_{\bar\pi(1)} \ldots a_{\bar\pi(l-k-1)}$, the position on the host cycle $C_{a_{\pi(0)} a_{\pi(1)} \ldots a_{\pi(k-1)}}$. [On each host cycle $C_{a_{\pi(0)} a_{\pi(1)} \ldots a_{\pi(k-1)}}$, $a_{\pi(0)}, a_{\pi(1)}, \ldots, a_{\pi(k-1)} \in \{0,1\}$, the same allocation function is used.] Formally, the embedding $f : V(CCC(l)) \to V(CCC(k))$ is of the form

$$f(i, a_0 a_1 \ldots a_{l-1}) := \big(d(i, a_{\bar\pi(0)} \ldots a_{\bar\pi(l-k-1)}), a_{\pi(0)} \ldots a_{\pi(k-1)}\big) \quad \text{for all } 0 \le i \le l-1,\ a_0 a_1 \ldots a_{l-1} \in \{0,1\}^l.$$

The load of f is determined by the allocation function d. Therefore, d should allocate the guest nodes as balancedly as possible on each host cycle. In the sequel, d will be chosen such that

$$\boxed{d(\pi(i), \beta) = i} \qquad \text{for all } 0 \le i \le k-1,\ \beta \in \{0,1\}^{l-k}.$$

This guarantees that all the cross-edges

$$(i,\alpha) \leftrightarrow (i,\alpha(i)), \quad i \in \{\pi(0),\pi(1),\dots,\pi(k-1)\},$$

of $CCC(l)$ are mapped onto a corresponding cross-edge in $CCC(k)$. All the other edges of $CCC(l)$ are mapped onto a path on a single cycle C_β in $CCC(k)$. So, in this case the dilation is directly dependent on the allocation d of the guest nodes on the host cycle and stands partly in contrast to the desired balancedness of the allocation as explained above.

For low dilation, the values of $\pi(0),\pi(1),\dots,\pi(k-1)$ should be allocated relatively balancedly among $0,1,\dots,l-1$, and the nodes $(i,a_0a_1\dots a_{l-1})$ and $(j,b_0b_1\dots b_{l-1})$ of the cycles $C_{\alpha_1},C_{\alpha_2},\dots,C_{\alpha_{2^{l-k}}}$ of $CCC(l)$ with a small lexicographical distance between $(i,a_{\bar\pi(0)}\dots a_{\bar\pi(l-k-1)})$ and $(j,b_{\bar\pi(0)}\dots b_{\bar\pi(l-k-1)})$ should be mapped close together on the cycle C_β in $CCC(k)$.

In [16], for $1 < l/k \le 2$, it was shown that the values of $\pi(0),\pi(1),\dots,\pi(k-1)$ can be specified such that the following holds:

a) $\pi(i+1)-\pi(i) \le 2$ for all $0 \le i < k-1$.

b) The nodes $\{(\pi(i),a_0a_1\dots a_{l-1}) \mid a_{\bar\pi(0)},a_{\bar\pi(1)},\dots,a_{\bar\pi(l-k-1)} \in \{0,1\}\}$ are mapped onto $(i,a_{\pi(0)}a_{\pi(1)}\dots a_{\pi(k-1)})$ for $0 \le i \le k-1$, $a_{\pi(0)},a_{\pi(1)},\dots,a_{\pi(k-1)} \in \{0,1\}$.

c) The nodes $\{(\bar\pi(i),a_0a_1\dots a_{l-1}) \mid 0 \le i \le l-k-1,\ a_{\bar\pi(0)},a_{\bar\pi(1)},\dots,a_{\bar\pi(l-k-1)} \in \{0,1\}\}$ can be allocated balancedly in certain sections of the host cycle $C_{a_{\pi(0)}a_{\pi(1)}\dots a_{\pi(k-1)}}$, $a_{\pi(0)},a_{\pi(1)},\dots,a_{\pi(k-1)} \in \{0,1\}$, while maintaining dilation 1 at the same time.

Here, for $\frac{5}{3}+c_k < \frac{l}{k} \le 2$, $c_k = \frac{4k+3}{3\cdot 2^{2/3k}}$, we show that $\pi(0),\pi(1),\dots,\pi(k-1)$ can be specified such that the following holds:

a) $\pi(i+1)-\pi(i) \le 3$ for all $0 \le i < k-1$.

b) The nodes $\{(\pi(i),a_0a_1\dots a_{l-1}) \mid a_{\bar\pi(0)},a_{\bar\pi(1)},\dots,a_{\bar\pi(l-k-1)} \in \{0,1\}\}$ are mapped onto $(i,a_{\pi(0)}a_{\pi(1)}\dots a_{\pi(k-1)})$ for $0 \le i \le k-1$, $a_{\pi(0)},a_{\pi(1)},\dots,a_{\pi(k-1)} \in \{0,1\}$.

c) The nodes $\{(\bar\pi(i),a_0a_1\dots a_{l-1}) \mid 0 \le i \le l-k-1,\ a_{\bar\pi(0)},a_{\bar\pi(1)},\dots,a_{\bar\pi(l-k-1)} \in \{0,1\}\}$ can be allocated balancedly on the complete host cycle $C_{a_{\pi(0)}a_{\pi(1)}\dots a_{\pi(k-1)}}$, $a_{\pi(0)},a_{\pi(1)},\dots,a_{\pi(k-1)} \in \{0,1\}$, while maintaining dilation 1 at the same time.

The main new technical contribution will be to show that the guest nodes $\{(\pi(i)+1,a_0a_1\dots a_{l-1}),(\pi(i)+2,a_0a_1\dots a_{l-1}) \mid a_{\bar\pi(0)},a_{\bar\pi(1)},\dots,a_{\bar\pi(l-k-1)} \in \{0,1\}\}$ can be allocated in an appropriate way among the host nodes $\{(j,a_{\pi(0)}a_{\pi(1)}\dots a_{\pi(k-1)}) \mid j \in \{i-1,i,i+1,i+2\}\}$ for $0 \le i < k-1$ such that $\pi(i+1)-\pi(i)=3$, while maintaining dilation 1 at the same time.

4 Improved Dilation 1 Embedding of the CCC

Theorem 1:

Let $k, l \in I\!N$, $k \geq 8$, such that $\frac{5}{3} + c_k < \frac{l}{k} \leq 2$, $c_k = \frac{4k+3}{3 \cdot 2^{2/3k}}$. Then, there is a dilation 1 embedding of $CCC(l)$ into $CCC(k)$ with load $\left\lceil \frac{l}{k} \cdot 2^{l-k} \right\rceil$.

Proof:

(A) $\boxed{l-k \text{ even}}$

We show that the construction of Section 3 can be adapted to yield an embedding of $CCC(l)$ into $CCC(k)$ with dilation 1 and load $\left\lceil \frac{l}{k} \cdot 2^{l-k} \right\rceil$.

For this, we specify the allocation d and the indices $\pi(i)$ for the embedding f in the construction of Section 3.

For $0 \leq i \leq \frac{l-k}{2} - 1$, let

$$h(i) := \left\lceil \frac{i \cdot 2l}{l-k} - \frac{3k}{2^{l-k}} \right\rceil + 1\,.$$

[Then, $h(0) = 1$, $h\left(\frac{l-k}{2} - 1\right) = l - 3\,$.] For $0 \leq i \leq l-k-1$, let

$$\bar{\pi}(i) := \begin{cases} h\left(\frac{i}{2}\right) & \text{if } i \text{ even,} \\ h\left(\left\lfloor \frac{i}{2} \right\rfloor\right) + 1 & \text{if } i \text{ odd.} \end{cases}$$

Let $\pi(0), \pi(1), \ldots, \pi(k-1) \in \{0, 1, \ldots, l-1\} \setminus \{\bar{\pi}(0), \bar{\pi}(1), \ldots, \bar{\pi}(l-k-1)\}$ such that $\pi(0) < \pi(1) < \ldots < \pi(k-1)$. [Note that $\{\pi(0), \pi(1), \ldots, \pi(k-1)\} \,\dot{\cup}\, \{\bar{\pi}(0), \bar{\pi}(1), \ldots, \bar{\pi}(l-k-1)\} = \{0, 1, \ldots, l-1\}$.]

For the time being, we only construct the allocation $d : \{0, 1, \ldots, l-1\} \times \{0,1\}^{l-k} \to \{0, 1, \ldots, k-1\}$ partially, namely we specify $d(i, \beta)$ for $i \in \{\pi(0), \pi(1), \ldots, \pi(k-1)\}$. Let

$$\boxed{d(\pi(i), \beta) := i} \qquad \text{for all } \; 0 \leq i \leq k-1,\; \beta \in \{0,1\}^{l-k}. \qquad (*)$$

[Later on, $d(i, \beta)$ is specified for $i \in \{\bar{\pi}(0), \bar{\pi}(1), \ldots, \bar{\pi}(l-k-1)\}$. For the moment, $d(i, \beta)$ may have an arbitrary value for $i \in \{\bar{\pi}(0), \bar{\pi}(1), \ldots, \bar{\pi}(l-k-1)\}$.]

Now, the embedding f of $CCC(l)$ into $CCC(k)$ is defined as in the construction of Section 3:

$$f(i, a_0 a_1 \ldots a_{l-1}) := (d(i, a_{\bar{\pi}(0)} \ldots a_{\bar{\pi}(l-k-1)}), a_{\pi(0)} \ldots a_{\pi(k-1)})$$
$$\text{for all } \; 0 \leq i \leq l-1,\; a_0 a_1 \ldots a_{l-1} \in \{0,1\}^l.$$

Note that (*) guarantees that all the cross-edges

$$(i,\alpha) \leftrightarrow (i,\alpha(i)), \quad i \in \{\pi(0),\pi(1),\ldots,\pi(k-1)\},$$

of $CCC(l)$ are mapped onto a corresponding cross-edge in $CCC(k)$ [see Claim A1 below].

Now, we construct $d(i,\beta)$ for $i \in \{\bar{\pi}(0),\bar{\pi}(1),\ldots,\bar{\pi}(l-k-1)\}$. Let $a_{\pi(0)}, a_{\pi(1)},\ldots,a_{\pi(k-1)} \in \{0,1\}$. For the time being, we allocate the guest nodes $\{(i, a_0a_1\ldots a_{l-1}) \mid i \in \{\bar{\pi}(0),\bar{\pi}(1),\ldots,\bar{\pi}(l-k-1)\},\ a_{\bar{\pi}(0)}, a_{\bar{\pi}(1)},\ldots,a_{\bar{\pi}(l-k-1)} \in \{0,1\}\}$ balancedly on the host cycle $C_{a_{\pi(0)}a_{\pi(1)}\cdots a_{\pi(k-1)}}$ according to the lexicographical order on $\{0,1,\ldots,l-1\}\times\{0,1\}^{l-k}$, i.e. we use an allocation function $\tilde{d} : \{\bar{\pi}(0),\bar{\pi}(1),\ldots,\bar{\pi}(l-k-1)\} \times \{0,1\}^{l-k} \to \{0,1,\ldots,k-1\}$ such that

- $\tilde{d}(\bar{\pi}(0), 0^{l-k}) = 0, \quad \tilde{d}(\bar{\pi}(l-k-1), 1^{l-k}) = k-1,$
- $\tilde{d}$ is monotonic nondecreasing in the lexicographic ordering of the arguments [i.e., $\tilde{d}(i,\beta) \le \tilde{d}(i',\beta')$, if $(i,\beta) \le (i',\beta')$ according to the lexicographical order on $\{0,1,\ldots,l-1\}\times\{0,1\}^{l-k}$],
- $\left\lceil \frac{l-k}{k}\cdot 2^{l-k}\right\rceil - 1 \ \le\ |\tilde{d}^{-1}(j)| \ \le\ \left\lceil \frac{l-k}{k}\cdot 2^{l-k}\right\rceil$ for all $j = 0,1,\ldots,k-1$.

[At this point, we are not concerned with the obtained dilation. We will see later on that the allocation $\tilde{d}$ can be changed into an allocation $d : \{\bar{\pi}(0),\bar{\pi}(1),\ldots,\bar{\pi}(l-k-1)\} \times \{0,1\}^{l-k} \to \{0,1,\ldots,k-1\}$ which guarantees dilation 1, while maintaining the balancedness of the allocation.]

Let $r(i) := h(i) - 2i - 1$ for all $0 \le i \le \frac{l-k}{2} - 1$. Then, according to Claim A2 below,

a) $\tilde{d}(h(i)-1,\beta) = r(i)$ for all $0 \le i \le \frac{l-k}{2} - 1,\ \beta \in \{0,1\}^{l-k}$,

b) $\tilde{d}(h(i)+2,\beta) = r(i)+1$ for all $0 \le i \le \frac{l-k}{2} - 1,\ \beta \in \{0,1\}^{l-k}$.

Also, according to Claim A3 below,

a) $r(i) - 1 \le \tilde{d}(h(i),\beta) \le \tilde{d}(h(i)+1,\beta) \le r(i)+2$
for all $0 \le i \le \frac{l-k}{2} - 1,\ \beta \in \{0,1\}^{l-k}$,

b) $|\tilde{d}^{-1}(r(i)-1) \cap \{(h(i),\beta),(h(i)+1,\beta) \mid \beta \in \{0,1\}^{l-k}\}| \ \le\ \left\lceil \frac{l-k}{k}\cdot 2^{l-k}\right\rceil - 1$
for all $0 \le i \le \frac{l-k}{2} - 1$,

c) $|\tilde{d}^{-1}(r(i)+2) \cap \{(h(i),\beta),(h(i)+1,\beta) \mid \beta \in \{0,1\}^{l-k}\}| \ \le\ \left\lceil \frac{l-k}{k}\cdot 2^{l-k}\right\rceil - 1$
for all $0 \le i \le \frac{l-k}{2} - 1$.

[As $h(i)-1, h(i)+2 \in \{\pi(0), \pi(1), \ldots, \pi(k-1)\}$, $h(i), h(i)+1 \in \{\bar{\pi}(0), \bar{\pi}(1), \ldots, \bar{\pi}(l-k-1)\}$, the dilation of the embedding f (using the allocation $\tilde{d}$ for $d(i,\beta)$, $i \in \{\bar{\pi}(0), \bar{\pi}(1), \ldots, \bar{\pi}(l-k-1)\}$) would be 2.]

Then, according to Claim A4 below, $\tilde{d}$ can be changed to an allocation d such that:

1.) Let $0 \le i \le \dfrac{l-k}{2} - 1$. For $1 \le j \le 4$, let

$$
\begin{aligned}
n_j &:= |d^{-1}(r(i)-2+j) \cap \{(h(i),\beta),(h(i)+1,\beta) \mid \beta \in \{0,1\}^{l-k}\}|\,,\\
\tilde{n}_j &:= |\tilde{d}^{-1}(r(i)-2+j) \cap \{(h(i),\beta),(h(i)+1,\beta) \mid \beta \in \{0,1\}^{l-k}\}|\,.
\end{aligned}
$$

Then,

$$
\begin{aligned}
& n_1 = \tilde{n}_1\,,\\
& n_2 \le \max\{\tilde{n}_2, \tilde{n}_1+1\} \quad \text{if } \tilde{n}_1 > 0\,,\\
& n_2 = \tilde{n}_2 \quad \text{if } \tilde{n}_1 = 0\,,\\
& n_3 \le \max\{\tilde{n}_3, \tilde{n}_4+1\} \quad \text{if } \tilde{n}_4 > 0\,,\\
& n_3 = \tilde{n}_3 \quad \text{if } \tilde{n}_4 = 0\,,\\
& n_4 = \tilde{n}_4\,.
\end{aligned}
$$

2.) For $0 \le i \le \dfrac{l-k}{2} - 1$, $\beta = b_{\bar{\pi}(0)} b_{\bar{\pi}(1)} \ldots b_{\bar{\pi}(l-k-1)} \in \{0,1\}^{l-k}$:

$$
\begin{aligned}
& r(i)-1 \le d(h(i),\beta) \le r(i)+1,\\
& r(i) \le d(h(i)+1,\beta) \le r(i)+2,\\
& |d(h(i)+1,\beta) - d(h(i),\beta)| \le 1,\\
& |d(h(i), b_{\bar{\pi}(0)} \ldots b_{\bar{\pi}(l-k-1)})\\
& \qquad - d(h(i), b_{\bar{\pi}(0)} \ldots b_{\bar{\pi}(2i-1)} \bar{b}_{\bar{\pi}(2i)} b_{\bar{\pi}(2i+1)} \ldots b_{\bar{\pi}(l-k-1)})| \le 1,\\
& |d(h(i)+1, b_{\bar{\pi}(0)} \ldots b_{\bar{\pi}(l-k-1)})\\
& \qquad - d(h(i)+1, b_{\bar{\pi}(0)} \ldots b_{\bar{\pi}(2i)} \bar{b}_{\bar{\pi}(2i+1)} b_{\bar{\pi}(2i+2)} \ldots b_{\bar{\pi}(l-k-1)})| \le 1.
\end{aligned}
$$

It follows that the final embedding f (using the allocation d) has dilation 1 and load $\left\lceil \dfrac{l}{k} \cdot 2^{l-k} \right\rceil$. □

CLAIM A1:

The cross-edge $(i,\alpha) \leftrightarrow (i,\alpha(i))$, $i = \pi(m)$, $0 \le m \le k-1$, of $CCC(l)$ is mapped by f onto a cross-edge in $CCC(k)$.

PROOF OF CLAIM A1:

f maps $(i,\alpha) = (i, a_0 a_1 \ldots a_{l-1})$ onto

$$(d(i, a_{\bar{\pi}(0)} \ldots a_{\bar{\pi}(l-k-1)}), a_{\pi(0)} \ldots a_{\pi(k-1)})$$

and $(i,\alpha(i))$ onto

$$(d(i, a_{\bar{\pi}(0)} \ldots a_{\bar{\pi}(l-k-1)}), a_{\pi(0)} \ldots a_{\pi(m-1)} \bar{a}_{\pi(m)} a_{\pi(m+1)} \ldots a_{\pi(k-1)})\,.$$

From (*),

$$d(i, a_{\bar{\pi}(0)} \dots a_{\bar{\pi}(l-k-1)}) = d(\pi(m), a_{\bar{\pi}(0)} \dots a_{\bar{\pi}(l-k-1)}) = m.$$

Hence, there is a cross-edge in $CCC(k)$ between the two image nodes of (i, α) and $(i, \alpha(i))$.

Claim A2:

$$\begin{array}{lll} a) & \tilde{d}(h(i)-1, \beta) = r(i) & \textit{for all } 0 \le i \le \dfrac{l-k}{2} - 1,\ \beta \in \{0,1\}^{l-k}, \\ b) & \tilde{d}(h(i)+2, \beta) = r(i)+1 & \textit{for all } 0 \le i \le \dfrac{l-k}{2} - 1,\ \beta \in \{0,1\}^{l-k}. \end{array}$$

Proof of Claim A2:

Proof of a):

First, notice that $h(i) - 1 = \pi(r)$ for some $r \in \{0, 1, \dots, k-1\}$. Hence, according to (*), it suffices to show that

$$r(i) = r\,.$$

For this purpose, observe that

$$\begin{aligned} r &= |\{0, 1, \dots, h(i)-1\} \cap \{\pi(0), \pi(1), \dots, \pi(k-1)\}| - 1 \\ &= h(i) - 1 - |\{0, 1, \dots, h(i)-1\} \cap \{\bar{\pi}(0), \bar{\pi}(1), \dots, \bar{\pi}(l-k-1)\}| \\ &= h(i) - 1 - 2i \\ &= r(i)\,. \end{aligned}$$

Proof of b):

First, notice that $h(i) + 2 = \pi(r)$ for some $r \in \{0, 1, \dots, k-1\}$. Hence, according to (*), it suffices to show that

$$r(i) + 1 = r\,.$$

For this purpose, observe that

$$\begin{aligned} r &= |\{0, 1, \dots, h(i)+2\} \cap \{\pi(0), \pi(1), \dots, \pi(k-1)\}| - 1 \\ &= h(i) + 2 - |\{0, 1, \dots, h(i)+2\} \cap \{\bar{\pi}(0), \bar{\pi}(1), \dots, \bar{\pi}(l-k-1)\}| \\ &= h(i) + 2 - (2i+2) \\ &= h(i) - 2i \\ &= r(i) + 1\,. \end{aligned}$$

CLAIM A3:

a) $r(i) - 1 \leq \tilde{d}(h(i), \beta)$

$$\text{for all} \;\; 0 \leq i \leq \frac{l-k}{2} - 1,\; \beta \in \{0,1\}^{l-k},$$

$$|\tilde{d}^{-1}(r(i) - 1) \cap \{(h(i), \beta), (h(i)+1, \beta) \mid \beta \in \{0,1\}^{l-k}\}| \;\leq\; \left\lceil \frac{l-k}{k} \cdot 2^{l-k} \right\rceil - 1$$

$$\text{for all} \;\; 0 \leq i \leq \frac{l-k}{2} - 1,$$

b) $\tilde{d}(h(i)+1, \beta) \leq r(i) + 2$

$$\text{for all} \;\; 0 \leq i \leq \frac{l-k}{2} - 1,\; \beta \in \{0,1\}^{l-k},$$

$$|\tilde{d}^{-1}(r(i) + 2) \cap \{(h(i), \beta), (h(i)+1, \beta) \mid \beta \in \{0,1\}^{l-k}\}| \;\leq\; \left\lceil \frac{l-k}{k} \cdot 2^{l-k} \right\rceil - 1$$

$$\text{for all} \;\; 0 \leq i \leq \frac{l-k}{2} - 1.$$

PROOF OF CLAIM A3:

Proof of a):

The number of nodes in $\{(\bar{\pi}(i), \beta) \mid 0 \leq i \leq l-k-1, \beta \in \{0,1\}^{l-k}\}$ between $(h(i), 0^{l-k})$ and $(\bar{\pi}(l-k-1), 1^{l-k})$ in lexicographical order is $(l-k-2i) \cdot 2^{l-k}$. If this number does not exceed the minimal capacity $(k - r(i) + 1) \cdot \left(\left\lceil \frac{l-k}{k} \cdot 2^{l-k} \right\rceil - 1 \right)$ of the nodes between $r(i) - 1$ and $k-1$, then a) follows. Hence, we have to check that

$$(l-k-2i) \cdot 2^{l-k} \;\leq\; (k - r(i) + 1) \cdot \left(\left\lceil \frac{l-k}{k} \cdot 2^{l-k} \right\rceil - 1 \right).$$

This is true, because

$$\begin{aligned}
&(k - r(i) + 1) \cdot \left(\left\lceil \frac{l-k}{k} \cdot 2^{l-k} \right\rceil - 1 \right) \\
&= \; (k - h(i) + 2i + 2) \cdot \left(\left\lceil \frac{l-k}{k} \cdot 2^{l-k} \right\rceil - 1 \right) \\
&= \; \left(k - \left\lceil \frac{i \cdot 2l}{l-k} - \frac{3k}{2^{l-k}} \right\rceil + 2i + 1 \right) \cdot \left(\left\lceil \frac{l-k}{k} \cdot 2^{l-k} \right\rceil - 1 \right) \\
&\geq \; \left(k - \left(\frac{i \cdot 2l}{l-k} - \frac{3k}{2^{l-k}} \right) + 2i \right) \cdot \left(\frac{l-k}{k} \cdot 2^{l-k} - 1 \right) \\
&= \; \left(k - \frac{i \cdot 2k}{l-k} + \frac{3k}{2^{l-k}} \right) \cdot \left(\frac{l-k}{k} \cdot 2^{l-k} - 1 \right) \\
&= \; (l-k-2i) \cdot 2^{l-k} + 3 \cdot \underbrace{(l-k)}_{\geq (\frac{2}{3} + c_k) \cdot k} - k + \underbrace{\frac{k}{l-k} \cdot 2i}_{\geq 0} - \underbrace{\frac{3k}{2^{l-k}}}_{l-k \geq 2/3k}
\end{aligned}$$

$$
\begin{aligned}
&\geq\ (l-k-2i)\cdot 2^{l-k}+k+\underbrace{k\cdot\frac{4k+3}{2^{2/3k}}-\frac{3k}{2^{2/3k}}}_{\geq 0}\\
&\geq\ (l-k-2i)\cdot 2^{l-k}+k\\
&\geq\ (l-k-2i)\cdot 2^{l-k}\,.
\end{aligned}
$$

Proof of b):
The number of nodes in $\{(\bar{\pi}(i),\beta)\mid 0\leq i\leq l-k-1,\beta\in\{0,1\}^{l-k}\}$ between $(h(0),0^{l-k})$ and $(h(i)+1,1^{l-k})$ in lexicographical order is $(2i+2)\cdot 2^{l-k}$. If this number does not exceed the minimal capacity $(r(i)+3)\cdot\left(\left\lceil\frac{l-k}{k}\cdot 2^{l-k}\right\rceil-1\right)$ of the nodes between 0 and $r(i)+2$, then b) follows. Hence, we have to check that

$$
(2i+2)\cdot 2^{l-k}\ \leq\ (r(i)+3)\cdot\left(\left\lceil\frac{l-k}{k}\cdot 2^{l-k}\right\rceil-1\right).
$$

This is true, because

$$
\begin{aligned}
&(r(i)+3)\cdot\left(\left\lceil\frac{l-k}{k}\cdot 2^{l-k}\right\rceil-1\right)\\
&=\ (h(i)-2i+2)\cdot\left(\left\lceil\frac{l-k}{k}\cdot 2^{l-k}\right\rceil-1\right)\\
&=\ \left(\left\lceil\frac{i\cdot 2l}{l-k}-\frac{3k}{2^{l-k}}\right\rceil-2i+3\right)\cdot\left(\left\lceil\frac{l-k}{k}\cdot 2^{l-k}\right\rceil-1\right)\\
&\geq\ \left(\frac{i\cdot 2l}{l-k}-\frac{3k}{2^{l-k}}-2i+3\right)\cdot\left(\frac{l-k}{k}\cdot 2^{l-k}-1\right)\\
&=\ \left(2i+3\cdot\underbrace{\frac{l-k}{k}}_{\geq\frac{2}{3}+c_k}\right)\cdot 2^{l-k}-3\cdot\underbrace{(l-k)}_{\leq k}-\underbrace{\frac{k}{l-k}\cdot 2i}_{\leq k}+\underbrace{\frac{3k}{2^{l-k}}}_{\geq 0}-3\\
&\geq\ \left(2i+3\cdot\left(\frac{2}{3}+\underbrace{\frac{4k+3}{3\cdot 2^{2/3k}}}_{2/3k\leq l-k}\right)\right)\cdot 2^{l-k}-4k-3\\
&\geq\ \left(2i+3\cdot\left(\frac{2}{3}+\frac{4k+3}{3\cdot 2^{l-k}}\right)\right)\cdot 2^{l-k}-4k-3\\
&=\ (2i+2)\cdot 2^{l-k}\,.
\end{aligned}
$$

Claim A4:

1.) *Let* $0 \leq i \leq \dfrac{l-k}{2} - 1$. *For* $1 \leq j \leq 4$, *let*

$$\begin{aligned} n_j &:= |d^{-1}(r(i)-2+j) \cap \{(h(i),\beta),(h(i)+1,\beta) \mid \beta \in \{0,1\}^{l-k}\}|\,, \\ \tilde{n}_j &:= |\tilde{d}^{-1}(r(i)-2+j) \cap \{(h(i),\beta),(h(i)+1,\beta) \mid \beta \in \{0,1\}^{l-k}\}|\,. \end{aligned}$$

Then,

$$\begin{aligned} &n_1 = \tilde{n}_1\,, \\ &n_2 \leq \max\{\tilde{n}_2, \tilde{n}_1+1\} \quad \text{if } \tilde{n}_1 > 0\,, \\ &n_2 = \tilde{n}_2 \quad \text{if } \tilde{n}_1 = 0\,, \\ &n_3 \leq \max\{\tilde{n}_3, \tilde{n}_4+1\} \quad \text{if } \tilde{n}_4 > 0\,, \\ &n_3 = \tilde{n}_3 \quad \text{if } \tilde{n}_4 = 0\,, \\ &n_4 = \tilde{n}_4\,. \end{aligned}$$

2.) *For* $0 \leq i \leq \dfrac{l-k}{2} - 1$, $\beta = b_{\bar{\pi}(0)} b_{\bar{\pi}(1)} \dots b_{\bar{\pi}(l-k-1)} \in \{0,1\}^{l-k}$ *:*

$$\begin{aligned} &r(i)-1 \leq d(h(i),\beta) \leq r(i)+1, \\ &r(i) \leq d(h(i)+1,\beta) \leq r(i)+2, \\ &|d(h(i)+1,\beta) - d(h(i),\beta)| \leq 1, \\ &|d(h(i), b_{\bar{\pi}(0)} \dots b_{\bar{\pi}(l-k-1)}) \\ &\qquad - d(h(i), b_{\bar{\pi}(0)} \dots b_{\bar{\pi}(2i-1)} \bar{b}_{\bar{\pi}(2i)} b_{\bar{\pi}(2i+1)} \dots b_{\bar{\pi}(l-k-1)})| \leq 1, \\ &|d(h(i)+1, b_{\bar{\pi}(0)} \dots b_{\bar{\pi}(l-k-1)}) \\ &\qquad - d(h(i)+1, b_{\bar{\pi}(0)} \dots b_{\bar{\pi}(2i)} \bar{b}_{\bar{\pi}(2i+1)} b_{\bar{\pi}(2i+2)} \dots b_{\bar{\pi}(l-k-1)})| \leq 1. \end{aligned}$$

Proof of Claim A4:

Let $n := l-k$. d is constructed as follows:

a) Consider the following lexicographical numbering on $\{0,1\}^n$:

$$\begin{aligned} &\mathrm{Lex}^1 : \{0,1\}^n \to \mathbb{N}_0\,, \\ &\mathrm{Lex}^1(b_0 b_1 \dots b_{n-1}) = b_0 2^{n-1} + b_1 2^{n-2} + \dots + b_{2i-1} 2^{n-2i} \\ &\qquad + b_{2i+2} 2^{n-2i-1} + b_{2i+3} 2^{n-2i-2} + \dots + b_{n-1} 2^2 \\ &\qquad + b_{2i+1} 2^1 + b_{2i} 2^0\,. \end{aligned}$$

Now, define:

$$\begin{aligned} &d(h(i),\beta) = r(i)-1 \quad \text{for all } \beta \in \{0,1\}^n,\ 0 \leq \mathrm{Lex}^1(\beta) \leq \tilde{n}_1 - 1, \\ &d(h(i)+1,\beta) = r(i) \quad \text{for all } \beta \in \{0,1\}^n,\ 0 \leq \mathrm{Lex}^1(\beta) \leq \tilde{n}_1 - 1. \\ &\text{If } \tilde{n}_1 \text{ odd: } d(h(i),\beta) = r(i) \quad \text{for all } \beta \in \{0,1\}^n,\ \mathrm{Lex}^1(\beta) = \tilde{n}_1. \end{aligned}$$

b) Consider the following lexicographical numbering on $\{0,1\}^n$:

$$\begin{aligned} &\mathrm{Lex}^2 : \{0,1\}^n \to \mathbb{N}_0\,, \\ &\mathrm{Lex}^2(b_0 b_1 \dots b_{n-1}) = b_0 2^{n-1} + b_1 2^{n-2} + \dots + b_{2i-1} 2^{n-2i} \\ &\qquad + b_{2i+2} 2^{n-2i-1} + b_{2i+3} 2^{n-2i-2} + \dots + b_{n-1} 2^2 \\ &\qquad + b_{2i} 2^1 + b_{2i+1} 2^0\,. \end{aligned}$$

Now, define:

$d(h(i),\beta) = r(i) + 2$
for all $\beta \in \{0,1\}^n$, $2^n - \tilde{n}_4 \leq \mathrm{Lex}^2(\beta) \leq 2^n - 1$,
$d(h(i)+1,\beta) = r(i) + 1$
for all $\beta \in \{0,1\}^n$, $2^n - \tilde{n}_4 \leq \mathrm{Lex}^2(\beta) \leq 2^n - 1$.
If $\tilde{n}_4$ odd: $d(h(i),\beta) = r(i) + 1$
for all $\beta \in \{0,1\}^n$, $\mathrm{Lex}^2(\beta) = 2^n - \tilde{n}_4 - 1$.

[Note that the definitions above are meaningful, because if we consider the lexicographical numbering

$$\mathrm{Lex}^3(b_0 b_1 \ldots b_{2i-1} b_{2i+2} \ldots b_{n-1}) = b_0 2^{n-3} + b_1 2^{n-4} + \ldots + b_{2i-1} 2^{n-2i-2} + b_{2i+2} 2^{n-2i-3} + b_{2i+3} 2^{n-2i-4} + \ldots + b_{n-1} 2^0$$

on $\{0,1\}^{n-2}$, in the first case, we choose the first $\left\lceil \frac{\tilde{n}_1}{4} \right\rceil$ bitstrings, and in the second case, we choose the last $\left\lceil \frac{\tilde{n}_4}{4} \right\rceil$ bitstrings. To guarantee that these bitstrings are different, we have to check that $\left\lceil \frac{\tilde{n}_1}{4} \right\rceil + \left\lceil \frac{\tilde{n}_4}{4} \right\rceil \leq 2^{n-2}$. But this is true for $\frac{l}{k} > \frac{5}{3}$.]

c) The nodes from $\{(h(i),\beta), (h(i)+1,\beta) \mid \beta \in \{0,1\}^n\}$ which have not been assigned a value for d in a) and b) are assigned arbitrary values from $\{r(i), r(i)+1\}$ such that the requirements for n_2 and n_3 are fulfilled.

From the definition of d, it follows immediately that d has the claimed properties 1.) and 2.) . $\square$

(B) $\boxed{l - k \text{ odd}}$

We show that the construction of Section 3 can be adapted to yield an embedding of $CCC(l)$ into $CCC(k)$ with dilation 1 and load $\left\lceil \frac{l}{k} \cdot 2^{l-k} \right\rceil$.

For this, we specify the allocation d and the indices $\pi(i)$ for the embedding f in the construction of Section 3.

For $0 \leq i \leq \frac{l-k-1}{2} - 1$, let

$$h(i) := \left\lceil \frac{i \cdot 2l}{l-k} - \frac{3k}{2^{l-k}} \right\rceil + 1 .$$

[Then, $h(0) = 1$, $h\left(\frac{l-k-1}{2} - 1\right) \in \{l-6, l-5\}$.] For $0 \leq i \leq l-k-2$, let

$$\bar{\pi}(i) := \begin{cases} h\left(\frac{i}{2}\right) & \text{if } i \text{ even,} \\ h\left(\left\lfloor \frac{i}{2} \right\rfloor\right) + 1 & \text{if } i \text{ odd.} \end{cases}$$

Let

$$\bar{\pi}(l-k-1) := l - 2 .$$

Let $\pi(0), \pi(1), \ldots, \pi(k-1) \in \{0,1,\ldots,l-1\} \setminus \{\bar{\pi}(0), \bar{\pi}(1), \ldots, \bar{\pi}(l-k-1)\}$ such that $\pi(0) < \pi(1) < \ldots < \pi(k-1)$. [Note that $\{\pi(0), \pi(1), \ldots, \pi(k-1)\} \,\dot{\cup}\, \{\bar{\pi}(0), \bar{\pi}(1), \ldots, \bar{\pi}(l-k-1)\} = \{0,1,\ldots,l-1\}$.]

For the time being, we only construct the allocation $d : \{0,1,\ldots,l-1\} \times \{0,1\}^{l-k} \to \{0,1,\ldots,k-1\}$ partially, namely we specify $d(i,\beta)$ for $i \in \{\pi(0), \pi(1), \ldots, \pi(k-1)\}$. Let

$$\boxed{d(\pi(i), \beta) := i} \qquad \text{for all } \ 0 \le i \le k-1,\ \beta \in \{0,1\}^{l-k}. \tag{*}$$

[Later on, $d(i,\beta)$ is specified for $i \in \{\bar{\pi}(0), \bar{\pi}(1), \ldots, \bar{\pi}(l-k-1)\}$. For the moment, $d(i,\beta)$ may have an arbitrary value for $i \in \{\bar{\pi}(0), \bar{\pi}(1), \ldots, \bar{\pi}(l-k-1)\}$.]

Now, the embedding f of $CCC(l)$ into $CCC(k)$ is defined as in the construction of Section 3:

$$\begin{aligned} f(i, a_0 a_1 \ldots a_{l-1}) &:= (d(i, a_{\bar{\pi}(0)} \ldots a_{\bar{\pi}(l-k-1)}), a_{\pi(0)} \ldots a_{\pi(k-1)}) \\ &\text{for all } \ 0 \le i \le l-1,\ a_0 a_1 \ldots a_{l-1} \in \{0,1\}^l. \end{aligned}$$

Note that (*) guarantees that all the cross-edges

$$(i,\alpha) \leftrightarrow (i, \alpha(i)), \quad i \in \{\pi(0), \pi(1), \ldots, \pi(k-1)\},$$

of $CCC(l)$ are mapped onto a corresponding cross-edge in $CCC(k)$ [see Claim B1 below].

Now, we construct $d(i,\beta)$ for $i \in \{\bar{\pi}(0), \bar{\pi}(1), \ldots, \bar{\pi}(l-k-1)\}$. Let $a_{\pi(0)}, a_{\pi(1)}, \ldots, a_{\pi(k-1)} \in \{0,1\}$. For the time being, we allocate the guest nodes $\{(i, a_0 a_1 \ldots a_{l-1}) \mid i \in \{\bar{\pi}(0), \bar{\pi}(1), \ldots, \bar{\pi}(l-k-1)\},\ a_{\bar{\pi}(0)}, a_{\bar{\pi}(1)}, \ldots, a_{\bar{\pi}(l-k-1)} \in \{0,1\}\}$ balancedly on the host cycle $C_{a_{\pi(0)} a_{\pi(1)} \cdots a_{\pi(k-1)}}$ according to the lexicographical order on $\{0,1,\ldots,l-1\} \times \{0,1\}^{l-k}$, i.e. we use an allocation function $\tilde{d} : \{\bar{\pi}(0), \bar{\pi}(1), \ldots, \bar{\pi}(l-k-1)\} \times \{0,1\}^{l-k} \to \{0,1,\ldots,k-1\}$ such that

- $\tilde{d}(\bar{\pi}(0), 0^{l-k}) = 0, \quad \tilde{d}(\bar{\pi}(l-k-1), 1^{l-k}) = k-1,$
- $\tilde{d}$ is monotonic nondecreasing in the lexicographic ordering of the arguments [i.e., $\tilde{d}(i,\beta) \le \tilde{d}(i',\beta')$, if $(i,\beta) \le (i',\beta')$ according to the lexicographical order on $\{0,1,\ldots,l-1\} \times \{0,1\}^{l-k}$],
- $\left\lceil \dfrac{l-k}{k} \cdot 2^{l-k} \right\rceil - 1 \ \le\ |\tilde{d}^{-1}(j)| \ \le\ \left\lceil \dfrac{l-k}{k} \cdot 2^{l-k} \right\rceil$ for all $j = 0,1,\ldots,k-1$.

[At this point, we are not concerned with the obtained dilation. We will see later on that the allocation $\tilde{d}$ can be changed into an allocation $d : \{\bar{\pi}(0), \bar{\pi}(1), \ldots, \bar{\pi}(l-k-1)\} \times \{0,1\}^{l-k} \to \{0,1,\ldots,k-1\}$ which guarantees dilation 1, while maintaining the balancedness of the allocation.]

Let $r(i) := h(i) - 2i - 1$ for all $0 \le i \le \dfrac{l-k-1}{2} - 1$. Then, according to Claim B2 below,

$$\begin{array}{lll}
\text{a)} & \tilde{d}(h(i)-1, \beta) = r(i) & \text{for all } 0 \le i \le \dfrac{l-k-1}{2} - 1,\ \beta \in \{0,1\}^{l-k}, \\
\text{b)} & \tilde{d}(h(i)+2, \beta) = r(i)+1 & \text{for all } 0 \le i \le \dfrac{l-k-1}{2} - 1,\ \beta \in \{0,1\}^{l-k}, \\
\text{c)} & \tilde{d}(\bar{\pi}(l-k-1)-1, \beta) = k-2 & \text{for all } \beta \in \{0,1\}^{l-k}, \\
\text{d)} & \tilde{d}(\bar{\pi}(l-k-1)+1, \beta) = k-1 & \text{for all } \beta \in \{0,1\}^{l-k}.
\end{array}$$

Also, according to Claim B3 below,

a) $r(i)-1 \le \tilde{d}(h(i),\beta) \le \tilde{d}(h(i)+1,\beta) \le r(i)+2$
for all $0 \le i \le \dfrac{l-k-1}{2}-1,\ \beta \in \{0,1\}^{l-k}$,

b) $|\tilde{d}^{-1}(r(i)-1) \cap \{(h(i),\beta),(h(i)+1,\beta) \mid \beta \in \{0,1\}^{l-k}\}| \ \le\ \left\lceil \dfrac{l-k}{k}\cdot 2^{l-k}\right\rceil - 1$
for all $0 \le i \le \dfrac{l-k-1}{2}-1$,

c) $|\tilde{d}^{-1}(r(i)+2) \cap \{(h(i),\beta),(h(i)+1,\beta) \mid \beta \in \{0,1\}^{l-k}\}| \ \le\ \left\lceil \dfrac{l-k}{k}\cdot 2^{l-k}\right\rceil - 1$
for all $0 \le i \le \dfrac{l-k-1}{2}-1$,

d) $k-2 \ \le \tilde{d}(\bar{\pi}(l-k-1),\beta) \ \le\ k-1$
for all $\beta \in \{0,1\}^{l-k}$.

[As $h(i)-1, h(i)+2, \bar{\pi}(l-k-1)-1, \bar{\pi}(l-k-1)+1 \in \{\pi(0),\pi(1),\ldots,\pi(k-1)\}$, $h(i), h(i)+1, \bar{\pi}(l-k-1) \in \{\bar{\pi}(0),\bar{\pi}(1),\ldots,\bar{\pi}(l-k-1)\}$, the dilation of the embedding f (using the allocation $\tilde{d}$ for $d(i,\beta)$, $i \in \{\bar{\pi}(0),\bar{\pi}(1),\ldots,\bar{\pi}(l-k-1)\}$) would be 2.]

Then, according to Claim B4 below, $\tilde{d}$ can be changed to an allocation d such that:

1.) Let $0 \le i \le \dfrac{l-k-1}{2}-1$. For $1 \le j \le 4$, let

$$n_j := |d^{-1}(r(i)-2+j) \cap \{(h(i),\beta),(h(i)+1,\beta) \mid \beta \in \{0,1\}^{l-k}\}|\,,$$
$$\tilde{n}_j := |\tilde{d}^{-1}(r(i)-2+j) \cap \{(h(i),\beta),(h(i)+1,\beta) \mid \beta \in \{0,1\}^{l-k}\}|\,.$$

Then,

$$\begin{array}{l}
n_1 = \tilde{n}_1\,,\\
n_2 \le \max\{\tilde{n}_2, \tilde{n}_1+1\} \quad \text{if } \ \tilde{n}_1 > 0\,,\\
n_2 = \tilde{n}_2 \quad \text{if } \ \tilde{n}_1 = 0\,,\\
n_3 \le \max\{\tilde{n}_3, \tilde{n}_4+1\} \quad \text{if } \ \tilde{n}_4 > 0\,,\\
n_3 = \tilde{n}_3 \quad \text{if } \ \tilde{n}_4 = 0\,,\\
n_4 = \tilde{n}_4\,.
\end{array}$$

2.) For $0 \le i \le \dfrac{l-k-1}{2}-1$, $\beta = b_{\bar{\pi}(0)}b_{\bar{\pi}(1)}\ldots b_{\bar{\pi}(l-k-1)} \in \{0,1\}^{l-k}$:

$$\begin{array}{l}
r(i)-1 \le d(h(i),\beta) \le r(i)+1,\\
r(i) \le d(h(i)+1,\beta) \le r(i)+2,\\
|d(h(i)+1,\beta) - d(h(i),\beta)| \le 1,\\
|d(h(i), b_{\bar{\pi}(0)}\ldots b_{\bar{\pi}(l-k-1)})\\
\qquad -d(h(i), b_{\bar{\pi}(0)}\ldots b_{\bar{\pi}(2i-1)}\bar{b}_{\bar{\pi}(2i)}b_{\bar{\pi}(2i+1)}\ldots b_{\bar{\pi}(l-k-1)})| \le 1,\\
|d(h(i)+1, b_{\bar{\pi}(0)}\ldots b_{\bar{\pi}(l-k-1)})\\
\qquad -d(h(i)+1, b_{\bar{\pi}(0)}\ldots b_{\bar{\pi}(2i)}\bar{b}_{\bar{\pi}(2i+1)}b_{\bar{\pi}(2i+2)}\ldots b_{\bar{\pi}(l-k-1)})| \le 1.
\end{array}$$

3.) For $k-2 \leq j \leq k-1$:

$$\begin{aligned} & |d^{-1}(j) \cap \{(\bar{\pi}(l-k-1), \beta) \mid \beta \in \{0,1\}^{l-k}\}| \\ = \;& |\tilde{d}^{-1}(j) \cap \{(\bar{\pi}(l-k-1), \beta) \mid \beta \in \{0,1\}^{l-k}\}|\,. \end{aligned}$$

For $\beta \in \{0,1\}^{l-k}$:

$$k-2 \leq d(\bar{\pi}(l-k-1), \beta) \leq k-1.$$

It follows that the final embedding f (using the allocation d) has dilation 1 and load $\left\lceil \frac{l}{k} \cdot 2^{l-k} \right\rceil$. □

Claim B1:

The cross-edge $(i, \alpha) \leftrightarrow (i, \alpha(i))$, $i = \pi(m)$, $0 \leq m \leq k-1$, of $CCC(l)$ is mapped by f onto a cross-edge in $CCC(k)$.

Proof of Claim B1:

Exactly like the proof of Claim A1.

Claim B2:

a) $\tilde{d}(h(i)-1, \beta) = r(i)$	*for all* $0 \leq i \leq \frac{l-k-1}{2} - 1$, $\beta \in \{0,1\}^{l-k}$,
b) $\tilde{d}(h(i)+2, \beta) = r(i)+1$	*for all* $0 \leq i \leq \frac{l-k-1}{2} - 1$, $\beta \in \{0,1\}^{l-k}$,
c) $\tilde{d}(\bar{\pi}(l-k-1)-1, \beta) = k-2$	*for all* $\beta \in \{0,1\}^{l-k}$,
d) $\tilde{d}(\bar{\pi}(l-k-1)+1, \beta) = k-1$	*for all* $\beta \in \{0,1\}^{l-k}$.

Proof of Claim B2:

The proof of a) and b) is exactly like the proof of Claim A2. c) and d) follow directly from the fact that $\bar{\pi}(l-k-1)-1 = \pi(k-2)$, $\bar{\pi}(l-k-1)+1 = \pi(k-1)$ and from (*).

Claim B3:

a) $r(i)-1 \leq \tilde{d}(h(i), \beta)$

for all $0 \leq i \leq \frac{l-k-1}{2} - 1$, $\beta \in \{0,1\}^{l-k}$,

$$|\tilde{d}^{-1}(r(i)-1) \cap \{(h(i), \beta), (h(i)+1, \beta) \mid \beta \in \{0,1\}^{l-k}\}| \leq \left\lceil \frac{l-k}{k} \cdot 2^{l-k} \right\rceil - 1$$

for all $0 \leq i \leq \frac{l-k-1}{2} - 1$, *b)* $\tilde{d}(h(i)+1,\beta) \leq r(i)+2$

for all $0 \leq i \leq \dfrac{l-k-1}{2} - 1,\ \beta \in \{0,1\}^{l-k}$,

$|\tilde{d}^{-1}(r(i)+2) \cap \{(h(i),\beta),(h(i)+1,\beta) \mid \beta \in \{0,1\}^{l-k}\}| \ \leq \ \left\lceil \dfrac{l-k}{k} \cdot 2^{l-k} \right\rceil - 1$

for all $0 \leq i \leq \dfrac{l-k-1}{2} - 1$,

c) $k-2 \ \leq \tilde{d}(\bar{\pi}(l-k-1),\beta) \ \leq \ k-1$
for all $\beta \in \{0,1\}^{l-k}$.

Proof of Claim B3:

The proof of a) and b) is exactly like the proof of Claim A3. c) follows directly from the definition of $\tilde{d}$ (and from the fact that $\dfrac{l}{k} \geq \dfrac{5}{3}$).

Claim B4:

1.) *Let* $0 \leq i \leq \dfrac{l-k-1}{2} - 1$. *For* $1 \leq j \leq 4$, *let*

$$n_j := \ |d^{-1}(r(i)-2+j) \cap \{(h(i),\beta),(h(i)+1,\beta) \mid \beta \in \{0,1\}^{l-k}\}|,$$
$$\tilde{n}_j := \ |\tilde{d}^{-1}(r(i)-2+j) \cap \{(h(i),\beta),(h(i)+1,\beta) \mid \beta \in \{0,1\}^{l-k}\}|.$$

Then,

$$\begin{aligned}
&n_1 = \tilde{n}_1\,,\\
&n_2 \leq \max\{\tilde{n}_2, \tilde{n}_1 + 1\} \quad \textit{if } \ \tilde{n}_1 > 0\,,\\
&n_2 = \tilde{n}_2 \quad \textit{if } \ \tilde{n}_1 = 0\,,\\
&n_3 \leq \max\{\tilde{n}_3, \tilde{n}_4 + 1\} \quad \textit{if } \ \tilde{n}_4 > 0\,,\\
&n_3 = \tilde{n}_3 \quad \textit{if } \ \tilde{n}_4 = 0\,,\\
&n_4 = \tilde{n}_4\,.
\end{aligned}$$

2.) *For* $0 \leq i \leq \dfrac{l-k-1}{2} - 1,\ \beta = b_{\bar{\pi}(0)} b_{\bar{\pi}(1)} \dots b_{\bar{\pi}(l-k-1)} \in \{0,1\}^{l-k}$:

$$\begin{aligned}
&r(i) - 1 \leq d(h(i),\beta) \leq r(i)+1,\\
&r(i) \leq d(h(i)+1,\beta) \leq r(i)+2,\\
&|d(h(i)+1,\beta) - d(h(i),\beta)| \leq 1,\\
&|d(h(i), b_{\bar{\pi}(0)} \dots b_{\bar{\pi}(l-k-1)})\\
&\qquad - d(h(i), b_{\bar{\pi}(0)} \dots b_{\bar{\pi}(2i-1)} \bar{b}_{\bar{\pi}(2i)} b_{\bar{\pi}(2i+1)} \dots b_{\bar{\pi}(l-k-1)})| \leq 1,\\
&|d(h(i)+1, b_{\bar{\pi}(0)} \dots b_{\bar{\pi}(l-k-1)})\\
&\qquad - d(h(i)+1, b_{\bar{\pi}(0)} \dots b_{\bar{\pi}(2i)} \bar{b}_{\bar{\pi}(2i+1)} b_{\bar{\pi}(2i+2)} \dots b_{\bar{\pi}(l-k-1)})| \leq 1.
\end{aligned}$$

3.) *For* $k-2 \leq j \leq k-1$:

$$\begin{aligned}
&|d^{-1}(j) \cap \{(\bar{\pi}(l-k-1),\beta) \mid \beta \in \{0,1\}^{l-k}\}|\\
= \quad &|\tilde{d}^{-1}(j) \cap \{(\bar{\pi}(l-k-1),\beta) \mid \beta \in \{0,1\}^{l-k}\}|\,.
\end{aligned}$$

For $\beta \in \{0,1\}^{l-k}$ *:*

$$k-2 \le d(\bar{\pi}(l-k-1),\beta) \le k-1.$$

PROOF OF CLAIM B4:

The proof of 1.) and 2.) is exactly like the proof of Claim A4. For 3.), define $d(\bar{\pi}(l-k-1),\beta) = \tilde{d}(\bar{\pi}(l-k-1),\beta)$ for all $\beta \in \{0,1\}^{l-k}$. Then, the claimed properties in 3.) follow immediately from the corresponding properties of $\tilde{d}$. □

5 Conclusion

In this paper, we have presented a new technique for the embedding of large cube-connected cycles networks into smaller ones. Using the new embedding strategy, we showed:

Let $k,l \in I\!N$, $k \ge 8$, *such that* $\frac{5}{3} + c_k < \frac{l}{k} \le 2$, $c_k = \frac{4k+3}{3 \cdot 2^{2/3k}}$. *Then, there is a dilation* 1 *embedding of* $CCC(l)$ *into* $CCC(k)$ *with load* $\left\lceil \frac{l}{k} \cdot 2^{l-k} \right\rceil$.

This is optimal, and improves the results from [16]. In the case that $\frac{l}{k} \cdot 2^{l-k} \in I\!N$, the embedding technique can be adapted to yield an even stronger result:

Let $k,l \in I\!N$ *such that* $\frac{5}{3} < \frac{l}{k} \le 2$, $\frac{l}{k} \cdot 2^{l-k} \in I\!N$. *Then, there is a dilation* 1 *embedding of* $CCC(l)$ *into* $CCC(k)$ *with load* $\left\lceil \frac{l}{k} \cdot 2^{l-k} \right\rceil$.

The embedding technique can also be applied in the case $\frac{3}{2} < \frac{l}{k} \le \frac{5}{3} + c_k$ yielding:

1. *Let* $k,l \in I\!N$, $k \ge 8$, *such that* $\frac{5}{3} < \frac{l}{k} \le \frac{5}{3} + c_k$, $c_k = \frac{4k+3}{3 \cdot 2^{2/3k}}$. *Then, there is a dilation* 1 *embedding of* $CCC(l)$ *into* $CCC(k)$ *with load* $\left\lceil \left(\frac{5}{3} + c_k\right) \cdot 2^{l-k} \right\rceil$.

2. *Let* $k,l \in I\!N$ *such that* $\frac{3}{2} < \frac{l}{k} < \frac{5}{3}$. *Let* $p \in \{1,2,\ldots\}$ *such that* $\frac{5p-4}{3p-2} < \frac{l}{k} \le \frac{5p+1}{3p+1}$. *Then, there is a dilation* 1 *embedding of* $CCC(l)$ *into* $CCC(k)$ *with load* $\left\lceil \frac{5p+1}{3p+1} \cdot 2^{l-k} \right\rceil$.

This also improves results from [16].

Unfortunately, the new embedding technique does not lead to any improvement in the case $1 < \frac{l}{k} \le \frac{3}{2}$. Hence, it is still of interest to improve the load of the non-optimal dilation 1 embeddings when $1 < \frac{l}{k} \le \frac{5}{3} + c_k$ (or to prove their optimality). Finally, a further study should also consider the congestion of the embeddings.

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