Inapproximability of Congestion Games

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Warwick 2007

Network Congestion Games

- Given a directed graph $G = (V, E)$ with delay functions $d_e : \{1, \ldots, n\} \rightarrow \mathbb{N}$, $e \in E$.
- Player i wants to allocate a path of minimal delay between a source s_i and a target t_i.

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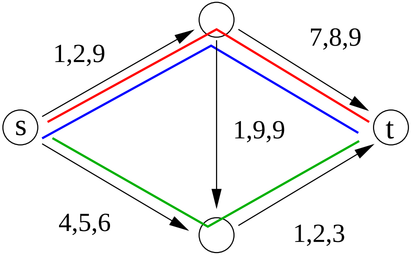

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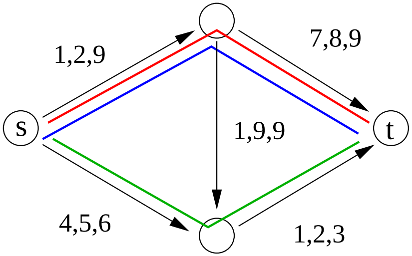

- Game is called *symmetric* if all players have the same source/target pair.

Congestion Games - general Definition

Congestion game is a tuple $G = (\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{N}}, (d_r)_{r \in \mathcal{R}})$ with

- $\mathcal{N} = \{1, \ldots, n\}$, set of players
- $\mathcal{R} = \{1, \ldots, m\}$, set of resources
- $\Sigma_i \subseteq 2^{[m]}$, strategy space of player i
- $d_r : \{1, \ldots, n\} \to \mathbb{R}$, delay function or resource r

For any state $S = (S_1, \ldots, S_n) \in \Sigma_1 \times \cdots \Sigma_n$

- n_r = number of players with $r \in S_i$
- $d_r(n_r)$ = delay of resource r
- $\sum_{r \in S_i} d_r(n_r)$ = delay of player i

S is Nash equilibrium if no player can unilaterally decrease its delay.

The transition graph

Definition

The *transition graph* of a congestion game Γ contains a node for every state S and a directed edge (S, S') if S' can be reached from S by the improvement step of a single player.

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Nash equilibria are local optima wrt *Rosenthal's potential function*

$$\phi(S) = \sum_{r \in \mathcal{R}} \sum_{i=1}^{n_r(S)} d_r(i) \ .$$

Complexity of computing equilibria

Known Results

	matroid games	network games	general games
symmetric	$O(n^2 m^2)$	polynomial	
asymmetric	$O(n^2 m^2)$		

matroid game results by [Ackermann, Röglin, V. 2006]
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The complexity class PLS

PLS (Polynomial Local Search)

PLS contains optimization problems with a specified neighborhood relationship Γ. It is required that there is a poly-time algorithm that, given any solution S,

- either computes a solution in $\Gamma(S)$ with better objective value
- or certifies that S is a local optimum.

Examples:

- FLIP (circuit evaluation with Flip-neighborhood)
- Max-Sat with Flip-neighborhood
- Max-Cut with Flip-neighborhood
- TSP with 2-Opt-neighborhood
- Congestion games wrt improvement steps

The complexity class PLS

PLS reductions

Given two PLS problems Π_1 and Π_2 find a mapping from the instances of Π_1 to the instances of Π_2 such that

- the mapping can be computed in polynomial time,
- the local optima of Π_1 are mapped to local optima of Π_2, and
- given any local optimum of Π_2, one can construct a local optimum of Π_1 in polynomial time.

Examples for PLS-complete problem:

- FLIP (via a master reduction)
- Max-Sat and POS-NAE-SAT
- Max-Cut

Approximation of local search problems

Definition

Consider any local search problem Π. Let $\alpha > 1$. An α-approximation for an instance of Π is a state S with the property that every state in $\Gamma(S)$ has a value of at most α times better than the value of S.

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Orlin, Punnen, Abraham, Schulz 2004

There is a fully polynomial time approximation scheme for every problem in PLS.

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Chien & Sinclair 2007

In any symmetric network congestion game in which all edges satisfy the β-bounded jump condition, i.e., $d_e(i+1) \leq \beta d_e(i)$ for all $i \in \mathbb{N}$, there is a sequence of improvement steps converging in

$$O(n\beta\epsilon^{-1}\log(nD))$$

steps, where D is an upper bound on the maximum delay.

New results

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New results

- For any poly-time computable $\alpha > 1$, finding an α-approximate Nash equilibrium in general congestion games with positive and increasing delay functions is PLS-hard.
- For every $n \in \mathbb{N}$, there is a congestion game with n players having a state with the property that every sequence of improvement steps leading from this state to an approximate equilibrium has exponential length in n.
- It is PSPACE-hard to compute an α-equilibrium reachable from a given state in a given congestion games.

Sketch of the analysis

We do a PLS-Reduction from FLIP.

Definition (FLIP)

An instance of the problem FLIP consists of a Boolean circuit C with input bits $x_1, \dots, x_n$ and output bits $y_1, \dots, y_m$. The neighborhood $N(x)$ of solution x is set of bit vectors x' that differ from x in one bit and $c(x') < c(x)$.

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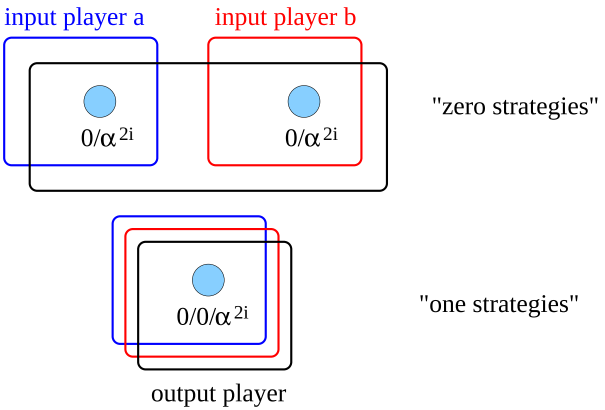
- We transform C into a congestion game $G(C)$ such that Nash equilibria of $G(C)$ correspond to a local optimum of f_C.
- Delays of different strategies of any player in $G(C)$ deviate at least by a factor of α.
- Thus all equilibria are α-approximate equilibria.

Representing circuits by congestion games

W.l.o.g. the circuit consists only of NAND-Gates. Sort the gates in reverse topological order. Design of i-th gate:



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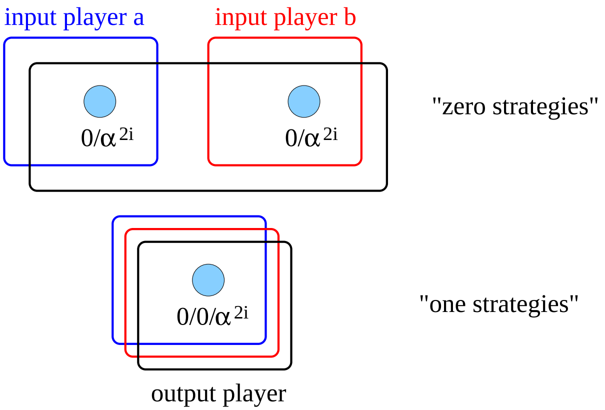

Input players "trigger" the output player because of the reverse topological order of the gates.

The feedback problem

A simple Idea that does not work out ...

1. Construct a circuit S with n input and n output bits that, for $x \in \{0,1\}^n$ not being a local optimum, computes $x' \in \{0,1\}^n$ with $f_C(x') \leq f_C(x)$.
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Problem: output players have much smaller delay differences than input players

PLS Reduction – High Level Description

Idea: Construct a congestion game simulating a processor

The players

- *n input players* $X_1, \ldots, X_n$
- *m clock players* $Z_1, \ldots, Z_m$ (with large delay differences)
- *gate players* for several circuits
- the *controller*

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Two kinds of states

Let M be a very large integer.

- In the *expensive states* at least one player has a delay of at least M.
- In the *inexpensive states* all players have a delay significantly smaller than M.

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- the controller together with the circuits guarantee the property $z \geq f_C(x)$ *(upper bound condition)*
- this ensures that the clock can only stop because it cannot trigger an improvement step
- consequently, every Nash equilibrium corresponds to a local optimum of C

Network Congestion Games

Conjecture

For any poly-time computable $\alpha > 1$, finding an α-approximate Nash equilibrium in network congestion games with positive and increasing delay functions is PLS-hard.

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Theorem

Let $\alpha, \beta > 1$ be appropriate constants. For every $n \in \mathbb{N}$, there is an n-player game with $O(n)$ resources and β-jump bounded delay function such that there is a state that has distance exponential in n to all α-Nash equilibria.

Open problems

- Is there a set of reasonable assumptions on the delay functions such that
  - computing an approximate Nash equilibrium has polynomial complexity?
  - improvement sequences reach an approximate Nash equilibrium after polynomially many steps?

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  - computing an approximate Nash equilibrium has polynomial complexity?
  - improvement sequences reach an approximate Nash equilibrium after polynomially many steps?
- What about other games? – e.g. the party affiliation game (Max-Cut)?