Local Distributed Decision

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Decision problems

Does randomization help?

Nondeterminism

Power of oracles

Further works
Outline

Decision problems

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Further works
Decide coloring
Computational model

**LOCAL model**

In each round during the execution of a distributed algorithm, every processor:

1. **sends** messages to its neighbors,
2. **receives** messages from its neighbors, and
3. **computes**, i.e., performs individual computations.

**Input**

An input configuration is a pair \((G, x)\) where \(G\) is a connected graph, and every node \(v \in V(G)\) is assigned as its local input a binary string \(x(v) \in \{0, 1\}^*\).

**Output**

The output of node \(v\) performing Algorithm \(A\) running in \(G\) with input \(x\) and identity assignment \(Id\):

\[\text{out}_A(G, x, Id, v)\]
Computational model

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The output of node \(v\) performing Algorithm \(\mathcal{A}\) running in \(G\) with input \(x\) and identity assignment \(\text{Id}\):

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\text{out}_\mathcal{A}(G, x, \text{Id}, v)
\]
Languages

A distributed language is a decidable collection of configurations.
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▶ Coloring = 
\{ (G, x) \text{ s.t. } \forall v \in V(G), \forall w \in N(v), x(v) \neq x(w) \}.
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- At-Most-One-Marked = \[ \{(G, x) \text{ s.t. } \| x \|_1 \leq 1\} \].
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- **Consensus** = \{(G, (x_1, x_2)) s.t. \( \exists u \in V(G), \forall v \in V(G), x_2(v) = x_1(u) \}\).
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- **MIS** = \( \{(G, x) \text{ s.t. } S = \{v \in V(G) \mid x(v) = 1\} \text{ is a MIS}\} \).
Decision

Let \( \mathcal{L} \) be a distributed language.

Algorithm \( \mathcal{A} \) decides \( \mathcal{L} \) \( \iff \) for every configuration \( (G, x) \):

- If \( (G, x) \in \mathcal{L} \), then for every identity assignment \( \text{Id} \),
  \( \text{out}_\mathcal{A}(G, x, \text{Id}, v) = \text{"yes"} \) for every node \( v \in V(G) \);
- If \( (G, x) \notin \mathcal{L} \), then for every identity assignment \( \text{Id} \),
  \( \text{out}_\mathcal{A}(G, x, \text{Id}, v) = \text{"no"} \) for at least one node \( v \in V(G) \).
Local decision

Let $t$ be a function of triplets $(G, x, \text{Id})$.

**Definition**

$\text{LD}(t)$ is the class of all distributed languages that can be decided by a distributed algorithm that runs in at most $t$ communication rounds.
Local decision

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**Definition**

$LD(t)$ is the class of all distributed languages that can be decided by a distributed algorithm that runs in at most $t$ communication rounds.

- Coloring $\in LD(1)$ and MIS $\in LD(1)$.
- AMOM, Consensus, and SpanningTree are not in LD($t$), for any $t = o(n)$. 
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Related work

What can be computed locally?
Define LCL as LD(O(1)) involving
- solely graphs of constant maximum degree
- inputs taken from a set of constant size

Theorem (Naor and Stockmeyer [STOC ’93])
If there exists a randomized algorithm that constructs a solution for a problem in LCL in O(1) rounds, then there is also a deterministic algorithm constructing a solution for that problem in O(1) rounds.

Proof uses Ramsey theory.

Not clearly extendable to languages in LD(O(1)) \ LCL.
$(\Delta + 1)$-coloring

**Arbitrary graphs**

- can be randomly computed in expected #rounds $O(\log n)$  
- best known deterministic algorithm performs in $2^{O(\sqrt{\log n})}$ rounds  
  (Panconesi, Srinivasan [J. Algorithms, 1996])

**Bounded degree graphs**

- Randomization does not help for 3-coloring the ring  
  (Naor [SIAM Disc. Maths 1991])
- can be randomly computed in expected #rounds $O(\log \Delta + \sqrt{\log n})$  
  (Schneider, Wattenhofer [PODC 2010])
- best known deterministic algorithm performs in $O(\Delta + \log^* n)$ rounds  
  (Barenboim, Elkin [STOC 2009]) (Kuhn [SPAA 2009])
Focus on distributed algorithms that use randomization but whose running time are deterministic.
2-sided error Monte Carlo algorithms

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\((p, q)\)-decider

- If \((G, x) \in \mathcal{L}\) then, for every identity assignment \(\text{Id}\),
  \[\Pr[\text{out}_A(G, x, \text{Id}, v) = \text{"yes" for every node } v \in V(G)] \geq p\]

- If \((G, x) \notin \mathcal{L}\) then, for every identity assignment \(\text{Id}\),
  \[\Pr[\text{out}_A(G, x, \text{Id}, v) = \text{"no" for at least one node } v \in V(G)] \geq q\]
Example: AMOM

Randomized algorithm
▶ every unmarked node says "yes" with probability 1;
▶ every marked node says "yes" with probability $p$.

Remarks:
▶ Runs in zero time;
▶ If the configuration has at most one marked node then correct with probability at least $p$.
▶ If there are at least $k \geq 2$ marked nodes, correct with probability at least $1 - p^k \geq 1 - p^2$.
▶ Thus there exists a $(p, q)$-decider for $q + p^2 \leq 1$. 
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Definition
BPLD\((t, p, q)\) is the class of all distributed languages that have a randomized distributed \((p, q)\)-decider running in time at most \(t\).
I.e., can be decided in time at most \(t\) by a randomized distributed algorithm with “yes” success probability \(p\) and “no” success probability \(q\).
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Remark

For \(p\) and \(q\) such that \(p^2 + q \leq 1\), there exists a language \(\mathcal{L} \in \text{BPLD}(0, p, q)\), such that \(\mathcal{L} \notin \text{LD}(t)\), for any \(t = o(n)\).
A sharp threshold for hereditary languages

A prefix of a configuration \((G, x)\) is a configuration \((G[U], x[U])\), where \(U \subseteq V(G)\)

Hereditary languages

A language \(L\) is hereditary if every prefix of every configuration \((G, x) \in L\) is also in \(L\).

- Coloring and AMOM are hereditary languages.
- Every language \(\{(G, \epsilon) \mid G \in G\}\) where \(G\) is hereditary is... hereditary. (Examples of hereditary graph families are planar graphs, interval graphs, forests, chordal graphs, cographs, perfect graphs, etc.)
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Theorem

Let \(\mathcal{L}\) be an hereditary language and let \(t\) be a function of triples \((G, x, ld)\). If \(\mathcal{L} \in BPLD(t, p, q)\) for constants \(p, q \in (0, 1]\) such that \(p^2 + q > 1\), then \(\mathcal{L} \in LD(O(t))\).
One ingredient in the proof

Let $0 < \delta < p^2 + q - 1$, and define $\lambda = 11 \cdot \lceil \log p / \log(1 - \delta) \rceil$.

Separating partition

A *separating partition* of $(G, x, \text{Id})$ is a triplet $(S, U_1, U_2)$ of pairwise disjoint subsets of nodes such that $S \cup U_1 \cup U_2 = V$, and $\text{dist}_G(U_1, U_2) \geq \lambda \cdot t$. 
Glueing lemma

Given a separating partition $\left( S, U_1, U_2 \right)$ of $\left( G, x, \text{Id} \right)$, let $G_k = G[U_k \cup S]$, and let $x_k$ be the input $x$ restricted to nodes in $G_k$, for $k = 1, 2$.

**Lemma ($\star$)**

For every instance $\left( G, x \right)$ with identity assignment $\text{Id}$, and every separating partition $\left( S, U_1, U_2 \right)$ of $\left( G, x, \text{Id} \right)$, we have:

$$\left( \left( G_1, x_1 \right) \in \mathcal{L} \text{ and } \left( G_2, x_2 \right) \in \mathcal{L} \right) \Rightarrow \left( G, x \right) \in \mathcal{L}.$$  

**Remark.** Lemma ($\star$) does not use the fact that $\mathcal{L}$ is hereditary, but uses $p^2 + q > 1$. 
Derandomization

**Deterministic Algorithm** \( \mathcal{D} \), applied at a node \( u \)

Given an instance \((G, x)\) and an id-assignment \( \text{Id} \):

\[
\text{If } (B_G(u, 2\lambda t), x[B_G(u, 2\lambda t)]) \in \mathcal{L} \\
\text{then } \text{out}(u) = \text{“yes”} \\
\text{else } \text{out}(u) = \text{“no”}
\]

**Remark**

Nodes do not know \( t \), but this can be fixed.
Correctness (1/4)

Assume \((G, x) \in \mathcal{L}\). Since \(\mathcal{L}\) is hereditary, every prefix of \((G, x)\) is also in \(\mathcal{L}\). Thus, every node \(u\) outputs \(\text{out}(u) = \text{"yes"}\).
Correctness (1/4)

Assume \((G, x) \in \mathcal{L}\).
Since \(\mathcal{L}\) is hereditary, every prefix of \((G, x)\) is also in \(\mathcal{L}\).
Thus, every node \(u\) outputs \(\text{out}(u) = \text{"yes"}\).

Conversely, assume \((G, x) \notin \mathcal{L}\).

Assume towards contradiction that by applying \(\mathcal{D}\) on \((G, x, \text{Id})\),
every node \(u\) outputs \(\text{out}(u) = \text{"yes"}\).

Let \(U \subseteq V(G)\) be a maximal set of vertices such that \(G[U]\) is
connected and \((G[U], x[U]) \in \mathcal{L}\).

\(\blacktriangleright\) \(U\) is not empty, as \((B_G(u, 2\lambda t), x[B_G(u, 2\lambda t)]) \in \mathcal{L}\) for every
node \(u\).

\(\blacktriangleright\) \(|U| < |V(G)|\), because \((G, x) \notin \mathcal{L}\).
Let $u \in U$ be a node such that $B_G(u, 2t)$ contains a node outside $U$.

Let $G' = G[U \cup V(B_G(u, 2t))]$.

Observe that $G'$ is connected and that $G'$ strictly contains $U$.

Our goal is to show that $(G', x[G']) \in \mathcal{L}$, for contradiction.
Correctness (3/4)

Let $W^1, W^2, \ldots, W^\ell$ be the $\ell$ connected components of $G[U] \setminus B_G(u, 2t)$, ordered arbitrarily.

Let $H$ denote the maximal graph such that $H$ is connected and

$$B_G(u, 2t) \subset V(H) \subseteq B_G(u, 2t) \cup (U \cap B_G(u, 2\lambda t))$$

Let $W^0$ be the empty graph, and for $k = 0, 1, 2, \ldots, \ell$, define the graph $Z^k = H \cup W^0 \cup W^1 \cup W^2 \cup \cdots \cup W^k$.

Observe that $Z^k$ is connected for each $k = 0, 1, 2, \ldots, \ell$, and that $Z^\ell = G'$.

We prove by induction on $k$ that $(Z^k, x[Z^k]) \in \mathcal{L}$ for every $k = 0, 1, 2, \ldots, \ell$. (This will establish the contradiction since, as we mentioned before, $Z^\ell = G'$).
Define the sets of nodes

- \( S = V(Z^k) \cap V(W^{k+1}) \)
- \( U_1 = V(Z^k) \setminus S \)
- \( U_2 = V(W^{k+1}) \setminus S \)

A crucial observation is that \((S, U_1, U_2)\) is a separating partition of \(Z^{k+1}\).

By induction, we have \((G_1, x[G_1]) \in \mathcal{L}\), because \(G_1 = G[U_1 \cup S] = Z^k\).

In addition, we have \((G_2, x[G_2]) \in \mathcal{L}\), because \(G_2 = G[U_2 \cup S] = W^{k+1}\), and \(W^{k+1}\) is a prefix of \(G[U]\).

We can now apply Lemma (⋆) and conclude that \((Z^{k+1}, x[Z^{k+1}]) \in \mathcal{L}\).
Outline

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Further works
Distributed certification

One motivation
Settings in which one must perform local verifications repeatedly.

- Proof Labeling Scheme (Korman, Kutten, Peleg [PODC 2007])
- Distributed verification (Das Sarma et al. [STOC 2011])
Distributed certification

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▶ Proof Labeling Scheme (Korman, Kutten, Peleg [PODC 2007])
▶ Distributed verification (Das Sarma et al. [STOC 2011])

Definition
An algorithm $\mathcal{A}$ verifies $\mathcal{L}$ if and only if for every configuration $(G, x)$, the following hold:

▶ If $(G, x) \in \mathcal{L}$, then there exists a certificate $y$ such that, for every id-assignment $\text{Id}$, $\text{out}_{\mathcal{A}}(G, (x, y), \text{Id}, v) = \text{"yes"}$ for all $v \in V(G)$;

▶ If $(G, x) \notin \mathcal{L}$, then for every certificate $y$, and for every id-assignment $\text{Id}$, $\text{out}_{\mathcal{A}}(G, (x, y), \text{Id}, v) = \text{"no"}$ for at least one node $v \in V(G)$. 
Non-determinism helps

**Definition**
Let $t$ be a function of triplets $(G, x, \text{Id})$. $\text{NLD}(t)$ is the class of all distributed languages that can be verified in at most $t$ communication rounds.

**Example**
$\text{Tree} = \{(G, \epsilon) \mid G \text{ is a tree}\} \in \text{NLD}(1)$.

Certificate given at node $v$ is $y(v) = \text{dist}_G(v, \hat{v})$, where $\hat{v} \in V(G)$ is an arbitrary fixed node.

Verification procedure verifies the following:

- $y(v)$ is a non-negative integer,
- if $y(v) = 0$, then $y(w) = 1$ for every neighbor $w$ of $v$, and
- if $y(v) > 0$, then there exists a neighbor $w$ of $v$ such that $y(w) = y(v) - 1$, and, for all other neighbors $w'$ of $v$, we have $y(w') = y(v) + 1$. 
NLD-complete problem

Reduction

$L_1$ is locally reducible to $L_2$, denoted by $L_1 \leq L_2$, if there exists a constant time local algorithm $A$ such that, for every configuration $(G, x)$ and every id-assignment $Id$, $A$ produces $\text{out}(v) \in \{0, 1\}^*$ as output at every node $v \in V(G)$ so that

$$(G, x) \in L_1 \iff (G, \text{out}) \in L_2.$$
NLD-complete problem

Reduction

$\mathcal{L}_1$ is locally reducible to $\mathcal{L}_2$, denoted by $\mathcal{L}_1 \preceq \mathcal{L}_2$, if there exists a constant time local algorithm $\mathcal{A}$ such that, for every configuration $(G, x)$ and every id-assignment $\text{Id}$, $\mathcal{A}$ produces $\text{out}(v) \in \{0, 1\}^*$ as output at every node $v \in V(G)$ so that

$$(G, x) \in \mathcal{L}_1 \iff (G, \text{out}) \in \mathcal{L}_2.$$ 

The language Containment

$x(v) = (E(v), S(v))$ where:

- $E(v)$ is an element
- $S(v)$ is a finite collection of sets

$\{(G, (E, S)) \mid \exists v \in V, \exists S \in S(v) \text{ s.t. } S \supseteq \{E(u) \mid u \in V\}\}.$
NLD-complete problem

Reduction
\( \mathcal{L}_1 \) is locally reducible to \( \mathcal{L}_2 \), denoted by \( \mathcal{L}_1 \leq \mathcal{L}_2 \), if there exists a constant time local algorithm \( A \) such that, for every configuration \( (G, x) \) and every id-assignment \( \text{Id} \), \( A \) produces \( \text{out}(v) \in \{0, 1\}^* \) as output at every node \( v \in V(G) \) so that

\[
(G, x) \in \mathcal{L}_1 \iff (G, \text{out}) \in \mathcal{L}_2.
\]

The language Containment
\( x(v) = (\mathcal{E}(v), S(v)) \) where:
- \( \mathcal{E}(v) \) is an element
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\{(G, (\mathcal{E}, S)) \mid \exists v \in V, \exists S \in S(v) \text{ s.t. } S \supseteq \{\mathcal{E}(u) \mid u \in V\}\}.
\]

Theorem
Containment is \( \text{NLD}(O(1)) \)-complete.
Combining non-determinism with randomization

Let \( \text{BPNLD}(t) = \bigcup_{p^2 + q \leq 1} \text{BPNLD}(t, p, q) \).
Combining non-determinism with randomization

Let $BPNLD(t) = \bigcup_{p^2 + q \leq 1} BPNLD(t, p, q)$.

**Theorem**

$BPNLD(O(1))$ contains all languages.
Combining non-determinism with randomization

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**Theorem**

$BPNLD(O(1))$ contains all languages.

**Proof**

The certificate is a map of the graph, i.e., an isomorphic copy $H$ of $G$, with nodes labeled from 1 to $n$.

Each node $v$ is also given its label $\ell(v)$ in $H$.

The proof that nodes can probabilistically check $H \sim G$ relies on two facts:

- To be “cheated”, a wrong map must be a lift of $G$.
- One can check whether $H$ is a lift of $G$ by having node(s) labeled 1 acting as in AMOM.
The “most difficult” decision problem

The problem \( \text{Cover} \)
\[ \{ (G, (\mathcal{E}, S)) \mid \exists v \in V, \exists S \in S(v) \text{ s.t. } S = \{ \mathcal{E}(u) \mid u \in V \} \}. \]

Theorem
\( \text{Cover is BPNLD}(O(1)) \)-complete.
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Numerous examples in the literature for which the knowledge of the size of the network is required to efficiently compute solutions.
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\[
\text{GraphSize} = \{ (G, k) \text{ s.t. } |V(G)| = k \}.
\]

**Theorem**

*For every language \( \mathcal{L} \), we have \( \mathcal{L} \in \mathcal{NLD}^{\text{GraphSize}} \).*

**Proof**

Certificate is the map of \( G \). (Cannot be “cheated” whenever the nodes know the number of nodes).
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- Also, one could restrict the memory used by a node, in addition to, or instead of, bounding the sequential time.
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- Also, one could restrict the memory used by a node, in addition to, or instead of, bounding the sequential time.
- Complexity framework taking also traffic congestion into account. (This can be done by, e.g., considering the CONGEST model).
Thank You!