# Local Distributed Decision

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# Outline

Decision problems

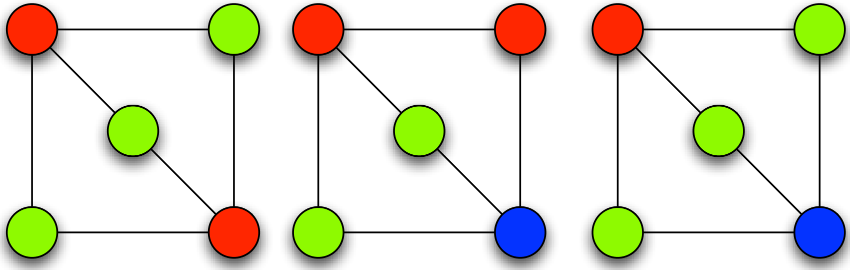
Does randomization helps?

Nondeterminism

Power of oracles

Further works

## Decide coloring



## Computational model

### $\mathcal{LOCAL}$ model

In each round during the execution of a distributed algorithm, every processor:

1. sends messages to its neighbors,
2. receives messages from its neighbors, and
3. computes, i.e., performs individual computations.

# Computational model

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## Input

An input configuration is a pair $(G, \mathrm{x})$ where $G$ is a connected graph, and every node $v \in V(G)$ is assigned as its local input a binary string $\mathrm{x}(v) \in \{0,1\}^*$.

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## Output

The output of node $v$ performing Algorithm $\mathcal{A}$ running in $G$ with input $\mathrm{x}$ and identity assignment $\mathrm{Id}$:

$$\mathrm{out}_{\mathcal{A}}(G, \mathrm{x}, \mathrm{Id}, v)$$

# Languages

A distributed language is a decidable collection of configurations.

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- `Consensus` $=$ $\{(G, (\mathrm{x}_1, \mathrm{x}_2))$ s.t. $\exists u \in V(G), \forall v \in V(G), \mathrm{x}_2(v) = \mathrm{x}_1(u)\}$.

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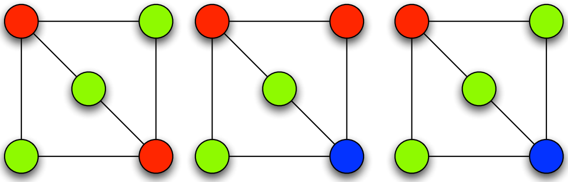
- `MIS` $= \{(G, \mathrm{x}) \text{ s.t. } S = \{v \in V(G) \mid \mathrm{x}(v) = 1\} \text{ is a MIS}\}$.

## Decision

Let $\mathcal{L}$ be a distributed language.

Algorithm $\mathcal{A}$ *decides* $\mathcal{L}$ $\iff$ for every configuration $(G, \mathrm{x})$:

- If $(G, \mathrm{x}) \in \mathcal{L}$, then for every identity assignment Id, $\mathrm{out}_{\mathcal{A}}(G, \mathrm{x}, \mathrm{Id}, v) =$ "yes" for every node $v \in V(G)$;
- If $(G, \mathrm{x}) \notin \mathcal{L}$, then for every identity assignment Id, $\mathrm{out}_{\mathcal{A}}(G, \mathrm{x}, \mathrm{Id}, v) =$"no" for at least one node $v \in V(G)$.



# Local decision

Let $t$ be a function of triplets $(G, \mathrm{x}, \mathrm{Id})$.

**Definition**

$\mathrm{LD}(t)$ is the class of all distributed languages that can be decided by a distributed algorithm that runs in at most $t$ communication rounds.

## Local decision

Let $t$ be a function of triplets $(G, \mathrm{x}, \mathrm{Id})$.

**Definition**
LD($t$) is the class of all distributed languages that can be decided by a distributed algorithm that runs in at most $t$ communication rounds.

- `Coloring` $\in$ LD(1) and `MIS` $\in$ LD(1).
- `AMOM`, `Consensus`, and `SpanningTree` are not in LD($t$), for any $t = o(n)$.

# Outline

# Related work

### What can be computed locally?

Define LCL as LD($O(1)$) involving

- solely graphs of constant maximum degree
- inputs taken from a set of constant size

**Theorem (Naor and Stockmeyer [STOC '93])**

*If there exists a randomized algorithm that constructs a solution for a problem in LCL in $O(1)$ rounds, then there is also a deterministic algorithm constructing a solution for that problem in $O(1)$ rounds.*

Proof uses Ramsey theory.

Not clearly extendable to languages in LD($O(1)$) $\setminus$ LCL.

# $(\Delta + 1)$-coloring

## Arbitrary graphs

- can be randomly computed in expected #rounds $O(\log n)$
  (Alon, Babai, Itai [J. Alg. 1986]) (Luby [SIAM J. Comput. 1986])
- best known deterministic algorithm performs in $2^{O(\sqrt{\log n})}$ rounds (Panconesi, Srinivasan [J. Algorithms, 1996])

## Bounded degree graphs

- Randomization does not help for 3-coloring the ring
  (Naor [SIAM Disc. Maths 1991])
- can be randomly computed in expected #rounds $O(\log \Delta + \sqrt{\log n})$ (Schneider, Wattenhofer [PODC 2010])
- best known deterministic algorithm performs in $O(\Delta + \log^* n)$ rounds
  (Barenboim, Elkin [STOC 2009]) (Kuhn [SPAA 2009])

## 2-sided error Monte Carlo algorithms

Focus on distributed algorithms that use randomization but whose running time are deterministic.

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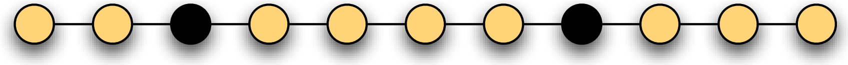
**$(p, q)$-decider**

- If $(G, \mathrm{x}) \in \mathcal{L}$ then, for every identity assignment Id, $\texttt{Pr}[\mathrm{out}_{\mathcal{A}}(G, \mathrm{x}, \mathrm{Id}, v) =$"yes" for every node $v \in V(G)] \geq p$
- If $(G, \mathrm{x}) \notin \mathcal{L}$ then, for every identity assignment Id, $\texttt{Pr}[\mathrm{out}_{\mathcal{A}}(G, \mathrm{x}, \mathrm{Id}, v) =$"no" for at least one node $v \in V(G)] \geq q$

## **Example:** AMOM



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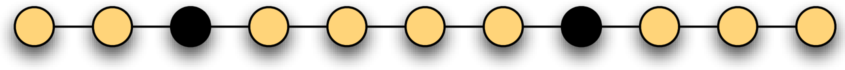

### Randomized algorithm

- every unmarked node says “yes” with probability 1;
- every marked node says “yes” with probability $p$.

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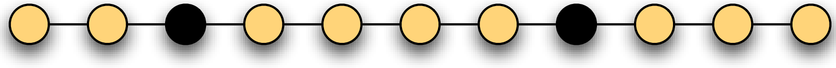

### Randomized algorithm

- every unmarked node says "yes" with probability 1;
- every marked node says "yes" with probability $p$.

### Remarks:

- Runs in zero time;
- If the configuration has at most one marked node then correct with probability at least $p$.
- If there are at least $k \geq 2$ marked nodes, correct with probability at least $1 - p^k \geq 1 - p^2$
- Thus there exists a $(p, q)$-decider for $q + p^2 \leq 1$.

# Bounded-probability error local decision

**Definition**
BPLD$(t, p, q)$ is the class of all distributed languages that have a randomized distributed $(p, q)$-decider running in time at most $t$.

I.e., can be decided in time at most $t$ by a randomized distributed algorithm with "yes" success probability $p$ and "no" success probability $q$.

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**Remark**
For $p$ and $q$ such that $p^2 + q \leq 1$, there exists a language $\mathcal{L} \in \mathrm{BPLD}(0, p, q)$, such that $\mathcal{L} \notin \mathrm{LD}(t)$, for any $t = o(n)$.

# A sharp threshold for hereditary languages

A prefix of a configuration $(G, \mathrm{x})$ is a configuration $(G[U], \mathrm{x}[U])$, where $U \subseteq V(G)$

## Hereditary languages

A language $\mathcal{L}$ is *hereditary* if every prefix of every configuration $(G, \mathrm{x}) \in \mathcal{L}$ is also in $\mathcal{L}$.

- `Coloring` and `AMOM` are hereditary languages.
- Every language $\{(G, \epsilon) \mid G \in \mathcal{G}\}$ where $\mathcal{G}$ is hereditary is... hereditary. (Examples of hereditary graph families are planar graphs, interval graphs, forests, chordal graphs, cographs, perfect graphs, etc.)

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### Theorem

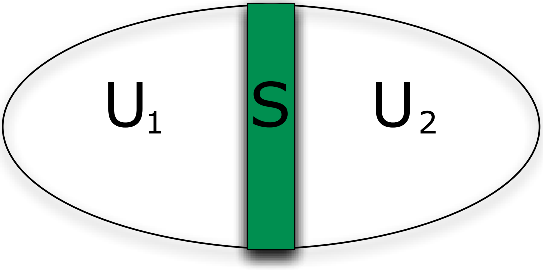
*Let $\mathcal{L}$ be an hereditary language and let $t$ be a function of triples $(G, x, Id)$. If $\mathcal{L} \in BPLD(t, p, q)$ for constants $p, q \in (0, 1]$ such that $p^2 + q > 1$, then $\mathcal{L} \in LD(O(t))$.*

## One ingredient in the proof

Let $0 < \delta < p^2 + q - 1$, and define $\lambda = 11 \cdot \lceil \log p / \log(1 - \delta) \rceil$.

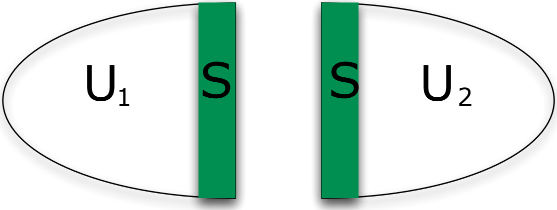
### Separating partition

A *separating partition* of $(G, \mathrm{x}, \mathrm{Id})$ is a triplet $(S, U_1, U_2)$ of pairwise disjoint subsets of nodes such that $S \cup U_1 \cup U_2 = V$, and $\mathrm{dist}_G(U_1, U_2) \geq \lambda \cdot t$.



## Glueing lemma

Given a separating partition $(S, U_1, U_2)$ of $(G, \mathrm{x}, \mathrm{Id})$, let $G_k = G[U_k \cup S]$, and let $\mathrm{x}_k$ be the input $\mathrm{x}$ restricted to nodes in $G_k$, for $k = 1, 2$.




**Lemma ($\star$)**

*For every instance $(G, x)$ with identity assignment $Id$, and every separating partition $(S, U_1, U_2)$ of $(G, x, Id)$, we have:*

$$\Big((G_1, x_1) \in \mathcal{L} \text{ and } (G_2, x_2) \in \mathcal{L}\Big) \;\Rightarrow\; (G, x) \in \mathcal{L}.$$

Remark. Lemma ($\star$) does not use the fact that $\mathcal{L}$ is hereditary, but uses $p^2 + q > 1$.

**Deterministic Algorithm $\mathcal{D}$, applied at a node $u$**

Given an instance $(G, \mathrm{x})$ and an id-assignment $\mathrm{Id}$:

$$\boxed{\begin{array}{l}\text{If } (B_G(u, 2\lambda t), \mathrm{x}[B_G(u, 2\lambda t)]) \in \mathcal{L} \\ \text{then } \mathrm{out}(u) = \text{“yes”} \\ \text{else } \mathrm{out}(u) = \text{“no”}\end{array}}$$

**Remark**

Nodes do not know $t$, but this can be fixed.

## Correctness (1/4)

Assume $(G, \mathrm{x}) \in \mathcal{L}$.
Since $\mathcal{L}$ is hereditary, every prefix of $(G, \mathrm{x})$ is also in $\mathcal{L}$.
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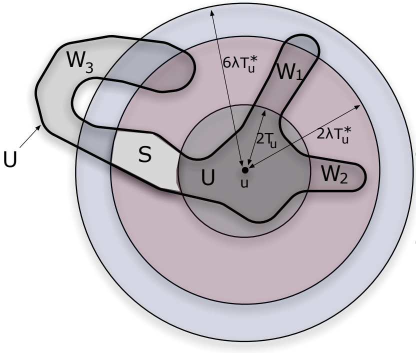
Conversely, assume $(G, \mathrm{x}) \notin \mathcal{L}$.

Assume towards contradiction that by applying $\mathcal{D}$ on $(G, \mathrm{x}, \mathrm{Id})$, every node $u$ outputs $\mathrm{out}(u) =$ “yes”.

Let $U \subseteq V(G)$ be a maximal set of vertices such that $G[U]$ is connected and $(G[U], \mathrm{x}[U]) \in \mathcal{L}$.

- $U$ is not empty, as $(B_G(u, 2\lambda t), \mathrm{x}[B_G(u, 2\lambda t)]) \in \mathcal{L}$ for every node $u$.
- $|U| < |V(G)|$, because $(G, \mathrm{x}) \notin \mathcal{L}$.

## Correctness (2/4)




Let $u \in U$ be a node such that $B_G(u, 2t)$ contains a node outside $U$.

Let $G' = G[U \cup V(B_G(u, 2t))]$.

Observe that $G'$ is connected and that $G'$ strictly contains $U$.

Our goal is to show that $(G', \mathrm{x}[G']) \in \mathcal{L}$, for contradiction.

## Correctness (3/4)

Let $W^1, W^2, \ldots, W^\ell$ be the $\ell$ connected components of $G[U] \setminus B_G(u, 2t)$, ordered arbitrarily.

Let $H$ denote the maximal graph such that $H$ is connected and

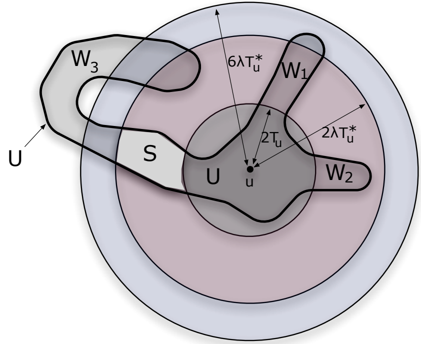
$$B_G(u, 2t) \subset \; V(H) \subseteq B_G(u, 2t) \cup (U \cap B_G(u, 2\lambda t))$$

Let $W^0$ be the empty graph, and for $k = 0, 1, 2, \cdots, \ell$, define the graph $Z^k \; = \; H \cup W^0 \cup W^1 \cup W^2 \cup \cdots \cup W^k$.

Observe that $Z^k$ is connected for each $k = 0, 1, 2, \cdots, \ell$, and that $Z^\ell = G'$.

We prove by induction on $k$ that $(Z^k, \mathrm{x}[Z^k]) \in \mathcal{L}$ for every $k = 0, 1, 2, \ldots, \ell$. (This will establish the contradiction since, as we mentioned before, $Z^\ell = G'$).

## Correctness (4/4)




Define the sets of nodes

- $S = V(Z^k) \cap V(W^{k+1})$
- $U_1 = V(Z^k) \setminus S$
- $U_2 = V(W^{k+1}) \setminus S$

A crucial observation is that $(S, U_1, U_2)$ is a separating partition of $Z^{k+1}$.

By induction, we have $(G_1, \mathrm{x}[G_1]) \in \mathcal{L}$, because $G_1 = G[U_1 \cup S] = Z^k$.

In addition, we have $(G_2, \mathrm{x}[G_2]) \in \mathcal{L}$, because $G_2 = G[U_2 \cup S] = W^{k+1}$, and $W^{k+1}$ is a prefix of $G[U]$.

We can now apply Lemma $(\star)$ and conclude that $(Z^{k+1}, \mathrm{x}[Z^{k+1}]) \in \mathcal{L}$.

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# Distributed certification

## One motivation

Settings in which one must perform local verifications repeatedly.

- Proof Labeling Scheme (Korman, Kutten, Peleg [PODC 2007])
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### Definition

An algorithm $\mathcal{A}$ verifies $\mathcal{L}$ if and only if for every configuration $(G, \mathrm{x})$, the following hold:

- If $(G, \mathrm{x}) \in \mathcal{L}$, then there exists a certificate $\mathrm{y}$ such that, for every id-assignment $\mathrm{Id}$, $\mathrm{out}_{\mathcal{A}}(G, (\mathrm{x}, \mathrm{y}), \mathrm{Id}, v) =$"yes" for all $v \in V(G)$;
- If $(G, \mathrm{x}) \notin \mathcal{L}$, then for every certificate $\mathrm{y}$, and for every id-assignment $\mathrm{Id}$, $\mathrm{out}_{\mathcal{A}}(G, (\mathrm{x}, \mathrm{y}), \mathrm{Id}, v) =$"no" for at least one node $v \in V(G)$.

# Non-determinism helps

### Definition

Let $t$ be a function of triplets $(G, \mathrm{x}, \mathrm{Id})$. $\mathrm{NLD}(t)$ is the class of all distributed languages that can be verified in at most $t$ communication rounds.

### Example

$\texttt{Tree} = \{(G, \epsilon) \mid G \text{ is a tree}\} \in \mathrm{NLD}(1)$.

Certificate given at node $v$ is $\mathrm{y}(v) = \mathrm{dist}_G(v, \hat{v})$, where $\hat{v} \in V(G)$ is an arbitrary fixed node.

Verification procedure verifies the following:

- $\mathrm{y}(v)$ is a non-negative integer,
- if $\mathrm{y}(v) = 0$, then $\mathrm{y}(w) = 1$ for every neighbor $w$ of $v$, and
- if $\mathrm{y}(v) > 0$, then there exists a neighbor $w$ of $v$ such that $\mathrm{y}(w) = \mathrm{y}(v) - 1$, and, for all other neighbors $w'$ of $v$, we have $\mathrm{y}(w') = \mathrm{y}(v) + 1$.

# NLD-complete problem

## Reduction

$\mathcal{L}_1$ is locally reducible to $\mathcal{L}_2$, denoted by $\mathcal{L}_1 \preceq \mathcal{L}_2$, if there exists a constant time local algorithm $\mathcal{A}$ such that, for every configuration $(G, \mathrm{x})$ and every id-assignment Id, $\mathcal{A}$ produces $\mathrm{out}(v) \in \{0,1\}^*$ as output at every node $v \in V(G)$ so that

$$(G, \mathrm{x}) \in \mathcal{L}_1 \iff (G, \mathrm{out}) \in \mathcal{L}_2 \ .$$

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## The language `Containment`

$x(v) = (\mathcal{E}(v), \mathcal{S}(v))$ where:

- $\mathcal{E}(v)$ is an element
- $\mathcal{S}(v)$ is a finite collection of sets

$\{(G, (\mathcal{E}, \mathcal{S})) \mid \exists v \in V, \ \exists S \in \mathcal{S}(v) \text{ s.t. } S \supseteq \{\mathcal{E}(u) \mid u \in V\}\}$.

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### Theorem

`Containment` *is $NLD(O(1))$-complete.*

## Combining non-determinism with randomization

Let $\mathrm{BPNLD}(t) = \cup_{p^2+q \leq 1} \mathrm{BPNLD}(t, p, q)$.

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**Proof**

The certificate is a map of the graph, i.e., an isomorphic copy $H$ of $G$, with nodes labeled from 1 to $n$.
Each node $v$ is also given its label $\ell(v)$ in $H$.
The proof that nodes can probabilistically check $H \sim G$ relies on two facts:

- To be "cheated", a wrong map must be a lift of $G$.
- One can check whether $H$ is a lift of $G$ by having node(s) labeled 1 acting as in AMOM.

□

## The “most difficult” decision problem

**The problem** `Cover`

$\{(G, (\mathcal{E}, \mathcal{S})) \mid \exists v \in V,\ \exists S \in \mathcal{S}(v) \text{ s.t. } S = \{\mathcal{E}(u) \mid u \in V\}\}.$

**Theorem**

`Cover` *is BPNLD(O(1))-complete.*

# Outline

# The oracle `GraphSize`

Numerous examples in the literature for which the knowledge of the size of the network is required to efficiently compute solutions.

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$$\texttt{GraphSize} = \{(G, k) \text{ s.t. } |V(G)| = k\}.$$

**Theorem**

*For every language $\mathcal{L}$, we have $\mathcal{L} \in NLD^{\texttt{GraphSize}}$.*

**Proof**

Certificate is the map of $G$. (Cannot be "cheated" whenever the nodes know the number of nodes).

# Outline

## Further works

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- Also, one could restrict the memory used by a node, in addition to, or instead of, bounding the sequential time.
- Complexity framework taking also traffic congestion into account. (This can be done by, e.g., considering the $\mathcal{CONGEST}$ model).

# Thank You!