

Approximate well-supported Nash equilibria in symmetric bimatrix games^{*}

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Abstract. The ε -well-supported Nash equilibrium is a strong notion of approximation of a Nash equilibrium, where no player has an incentive greater than ε to deviate from any of the pure strategies that she uses in her mixed strategy. The smallest constant ε currently known for which there is a polynomial-time algorithm that computes an ε -well-supported Nash equilibrium in bimatrix games is slightly below $2/3$. In this paper we study this problem for *symmetric* bimatrix games and we provide a polynomial-time algorithm that gives a $(1/2 + \delta)$ -well-supported Nash equilibrium, for an arbitrarily small positive constant δ .

1 Introduction

The problem of computing Nash equilibria is one of the most fundamental problems in algorithmic game theory. It is now known that the complexity of computing a Nash equilibrium is PPAD-complete [4], even for two-player games [3]. Given this evidence of intractability of the problem, further research has focused on the computation of *approximate* Nash equilibria. In this context—and assuming that all payoffs are normalized to be in the interval $[0, 1]$ —the standard notion of approximation is the additive approximation with a parameter $\varepsilon \in [0, 1]$. There are two different notions of additive approximation of Nash equilibria: the ε -Nash equilibrium and the ε -well-supported Nash equilibrium.

An ε -Nash equilibrium is a strategy profile—one strategy for each player—in which no player can improve her payoff by more than ε through unilateral deviation from her strategy in the strategy profile. Several polynomial-time algorithms have been proposed to find ε -Nash equilibria for $\varepsilon = 1/2$ [6], for $\varepsilon = (3 - \sqrt{5})/2 \approx 0.38$ [5], for $\varepsilon = 1/2 - 1/(3\sqrt{6}) \approx 0.36$ [2], and finally for $\varepsilon \approx 0.3393$ [13]. It is also known how to find ε -Nash equilibria in quasi-polynomial time $n^{O(\log n/\varepsilon^2)}$ for arbitrarily small $\varepsilon > 0$ [11], where n is the number of pure strategies.

The notion of an ε -well-supported Nash equilibrium requires that no player has an incentive greater than ε to deviate from any of the pure strategies she uses in her mixed strategy. It is a notion stronger than that of an ε -Nash equilibrium:

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every ε -well-supported Nash equilibrium is also an ε -Nash equilibrium, but not necessarily vice-versa. The smallest ε for which a polynomial-time algorithm is currently known that computes an ε -well-supported Nash equilibrium in an arbitrary bimatrix game is slightly above 0.6619 [9, 7]. It is also known that for the class of win-lose bimatrix games one can find $1/2$ -well-supported Nash equilibria in polynomial time [9].

In this paper we study computation of approximate well-supported Nash equilibria in *symmetric* bimatrix games, a class of bimatrix games in which swapping the roles of the two players does not change the payoff matrices, that is if the payoff matrix of one is the transpose of the payoff matrix of the other. Symmetric games are an important class of games in game theory; their applications include auctions and congestion games. They have already been studied by Nash in his seminal paper in which he introduced the concept of a Nash equilibrium; he proved that every symmetric game has at least one symmetric Nash equilibrium, that is one in which all players use the same mixed strategy [12].

Computing Nash equilibria in symmetric bimatrix games is known to be as hard as computing Nash equilibria in arbitrary bimatrix games because there is a polynomial-time reduction from the latter to the former [8]. In contrast to arbitrary bimatrix games, it is known how to compute $(1/3 + \delta)$ -Nash equilibria in symmetric bimatrix games in polynomial time, where $\delta > 0$ is arbitrarily small [10]. In this paper we improve our understanding of the approximability of Nash equilibria in symmetric bimatrix games by considering the task of computing approximate well-supported Nash equilibria. Our main result is an algorithm that computes $(1/2 + \delta)$ -well-supported Nash equilibria in symmetric bimatrix games in polynomial time, where $\delta > 0$ is arbitrarily small (Theorem 3).

Our $(1/2 + \delta)$ -approximation algorithm splits the analysis into two cases that are then considered independently. The first case is based on the following relaxation of the concept of a symmetric Nash equilibrium: we say that a strategy profile (x, x) *prevents exceeding* $u \in [0, 1]$ if the expected payoff of every pure strategy in the symmetric game is at most u when the other player uses strategy x . This is indeed a relaxation of the concept of the symmetric Nash equilibrium because if (x^*, x^*) is a symmetric Nash equilibrium then it prevents exceeding its value (that is, the expected payoff each player gets when they both play strategy x^*). We justify relevance of this concept by showing that a strategy profile (x, x) that prevents exceeding u is a u -well-supported Nash equilibrium, so in order to provide a latter it is sufficient to find a former. Moreover, we show that this relaxation of a symmetric Nash equilibrium is algorithmically tractable because it suffices to solve a single linear program to find a strategy profile (x, x) that prevents exceeding u , if there is one. The first case in our algorithm is to solve this linear program for $u = 1/2$ and if it succeeds then we can immediately report a $1/2$ -well-supported Nash equilibrium. Note that by the above, if there is indeed a symmetric Nash equilibrium with value $1/2$ or smaller, then the linear program does have a solution.

If the first case in the algorithm fails to identify a $1/2$ -well-supported equilibrium because the game has no symmetric Nash equilibrium with value $1/2$

or smaller, then we consider the other, and technically more challenging, case. We use another relaxation of the concept of a symmetric Nash equilibrium: we say that a strategy profile (x, y) *well supports* $u \in [0, 1]$ if the expected payoff of every pure strategy in the support of x is at least u when the other player uses strategy y , and the expected payoff of every pure strategy in the support of y is at least u when the other player uses strategy x . We observe that if a strategy profile (x, y) well supports u then it is a $(1 - u)$ -well-supported Nash equilibrium, so in order to provide a latter it is sufficient to find a former.

Therefore, in order to obtain a $(1/2 + \delta)$ -well-supported Nash equilibrium, we are interested in finding a strategy profile (x, y) that well supports $u \geq 1/2 - \delta$. While it may not be easy to verify if there is such a strategy profile, let alone find one, both can be achieved in polynomial time by solving a single linear program if we happen to know the supports of strategies of each player in such a strategy profile. The obvious technical obstacle to algorithmic tractability here is that the number of all possible supports to consider is exponential in the number of pure strategies. We overcome this difficulty by proving the main technical result of the paper (Theorem 2) that for every symmetric Nash equilibrium (x^*, x^*) and for every $\delta > 0$ establishes existence of a strategy profile (x, y) , with both strategies having supports of constant size, that well supports $u^* - \delta$, where u^* is the value of the Nash equilibrium. Note that by the failure of the first case every symmetric Nash equilibrium has value larger than $1/2$, and hence Theorem 2 implies that there is such a strategy profile with constant-size supports that well supports $1/2 - \delta$. The second case of our algorithm is to solve the linear programs mentioned above for $u = 1/2 - \delta$ and for all supports I and J of sizes at most $\kappa(\delta)$ —where $\kappa(\delta)$ is a constant (which depends on δ , but does not depend on the number n of pure strategies) that is specified in Theorem 2—and to output a solution (x, y) as soon as one is found.

In order to prove our main technical result (Theorem 2) we use the probabilistic method to prove existence of constant-support strategy profiles that nearly well support the expected payoffs of a Nash equilibrium. Our construction and proof are inspired by the construction of Daskalakis et al. [5] used by them to compute $(3 - \sqrt{5})/2$ -Nash equilibria in bimatrix games in polynomial time, but our analysis is different and more involved because we need to guarantee the extra condition of nearly well supporting the equilibrium values. The general idea of using sampling and Hoeffding bounds to prove existence of approximate equilibria with small supports dates back to the papers of Althofer [1] and Lipton et al. [11], who have shown that strategies with supports of size $O(\log n / \varepsilon^2)$ are sufficient for ε -Nash equilibria in games with n strategies.

2 Preliminaries

We consider bimatrix games (R, C) , where $R, C \in [0, 1]^{n \times n}$ are square matrices of payoffs for the two players: the row player and the column player, respectively. If the row player uses a strategy i , $1 \leq i \leq n$ and if the column one uses a strategy j , $1 \leq j \leq n$, then the row player receives payoff R_{ij} and the column player

receives payoff C_{ij} . We assume that the payoff values are in the interval $[0, 1]$; it is easy to see that equilibria in bimatrix games are invariant under additive and positive multiplicative transformations of the payoff matrices.

A *mixed strategy* $x \in [0, 1]^n$ is a probability distribution on the set of *pure strategies* $\{1, 2, \dots, n\}$. If the row player uses a mixed strategy x and the column player uses a mixed strategy y , then the row player receives payoff $x^T R y$ and the column player receives payoff $x^T C y$. A pair of strategies (x, y) , the former for the row player and the latter for the column player, is often referred to as a strategy profile. We define the *support* $\text{supp}(x)$ of a mixed strategy x to be the set of pure strategies that have positive probability in x , i.e., $\text{supp}(x) = \{i : 1 \leq i \leq n \text{ and } x_i > 0\}$.

For every i , $1 \leq i \leq n$, let $R_{i\bullet}$ be the row vector of the payoffs of the payoff matrix R when the row player uses the strategy i . Note that if the row player uses a pure strategy i , $1 \leq i \leq n$, and if the column player uses a mixed strategy y , then the row player receives payoff $R_{i\bullet} y$. Similarly, for every j , $1 \leq j \leq n$, let $C_{\bullet j}$ be the column vector of the payoffs of the matrix C when the column player uses the strategy j . Note that if the column player uses a pure strategy j , $1 \leq j \leq n$, and if the row player uses a mixed strategy x , then the column player receives payoff $x^T C_{\bullet j}$.

Definition 1 (Nash equilibrium). A Nash equilibrium is a strategy profile (x^*, y^*) such that

- for every i , $1 \leq i \leq n$, we have $R_{i\bullet} y^* \leq (x^*)^T R y^*$, and
- for every j , $1 \leq j \leq n$, we have $(x^*)^T C_{\bullet j} \leq (x^*)^T C y^*$,

or, in other words, if x^* is a best response to y^* and y^* is a best response to x^* .

Definition 2 (Approximate Nash equilibrium). For every $\varepsilon > 0$, an ε -Nash equilibrium is a strategy profile (x^*, y^*) such that

- for every i , $1 \leq i \leq n$, we have $R_{i\bullet} y^* - (x^*)^T R y^* \leq \varepsilon$, and
- for every j , $1 \leq j \leq n$, we have $(x^*)^T C_{\bullet j} - (x^*)^T C y^* \leq \varepsilon$,

or, in other words, if x^* is an ε -best response to y^* and y^* is an ε -best response to x^* .

Definition 3 (Approximate well-supported Nash equilibrium). For every $\varepsilon > 0$, an ε -well-supported Nash equilibrium is a strategy profile (x^*, y^*) such that

- for every i , $1 \leq i \leq n$, and $i' \in \text{supp}(x^*)$, we have $R_{i\bullet} y^* - R_{i'\bullet} y^* \leq \varepsilon$, and
- for every j , $1 \leq j \leq n$, and $j' \in \text{supp}(y^*)$, we have $(x^*)^T C_{\bullet j} - (x^*)^T C_{\bullet j'} \leq \varepsilon$,

or, in other words, if every $i' \in \text{supp}(x^*)$ is an ε -best response to y^* and every $j' \in \text{supp}(y^*)$ is an ε -best response to x^* .

Definition 4 (Symmetric game, symmetric Nash equilibrium). A bimatrix game (R, C) is symmetric if $C = R^T$.

A symmetric Nash equilibrium in a symmetric bimatrix game (R, R^T) is a strategy profile (x^*, x^*) such that for every i , $1 \leq i \leq n$, we have $R_{i\bullet}x^* \leq (x^*)^T R x^*$. Note that then it also follows that for every j , $1 \leq j \leq n$, we have:

$$(x^*)^T R_{\bullet j}^T = R_{j\bullet}x^* \leq (x^*)^T R x^* = (R x^*)^T x^* = (x^*)^T R^T x^*.$$

Let us recall a fundamental theorem of Nash [12] about existence of symmetric Nash equilibria in symmetric bimatrix games.

Theorem 1 ([12]). Every symmetric bimatrix game has a symmetric Nash equilibrium.

3 Computing approximate well-supported Nash equilibria

Fix a bimatrix game $G = (R, C)$ for the rest of the paper, where $R, C \in [0, 1]^{n \times n}$. We will use N to denote the number of bits needed to represent the matrices R and C with all their entries represented in binary. We say that a strategy x is k -uniform, for $k \in \mathbb{N} \setminus \{0\}$, if $x_i \in \{0, \frac{1}{k}, \frac{2}{k}, \dots, 1\}$, for every i , $1 \leq i \leq n$.

3.1 Strategies that prevent exceeding a payoff

Definition 5 (Preventing exceeding payoffs). We say that a strategy $x \in [0, 1]^n$ for the row player prevents exceeding $u \in [0, 1]$ if for every $j = 1, 2, \dots, n$, we have $x^T C_{\bullet j} \leq u$ or, in other words, if the column player payoff of the best response to x does not exceed u . Similarly, we say that a strategy $y \in [0, 1]^n$ for the column player prevents exceeding $v \in [0, 1]$ if for every $i = 1, 2, \dots, n$, we have $R_{i\bullet}y \leq v$ or, in other words, if the row player payoff of the best response to y does not exceed v .

For brevity, we say that a strategy profile (x, y) prevents exceeding (v, u) if x prevents exceeding u and y prevents exceeding v .

Observe that the following system of linear constraints $\text{PE}(v, u)$ characterizes strategy profiles (x, y) that prevent exceeding $(v, u) \in [0, 1]^2$:

$$\begin{aligned} \sum_{i=1}^n x_i &= 1; \quad x_i \geq 0 \text{ for all } i = 1, 2, \dots, n; \\ \sum_{j=1}^n y_j &= 1; \quad y_j \geq 0 \text{ for all } j = 1, 2, \dots, n; \\ R_{i\bullet}y &\leq v \text{ for all } i = 1, 2, \dots, n; \\ x^T C_{\bullet j} &\leq u \text{ for all } j = 1, 2, \dots, n. \end{aligned}$$

Note that if (x, y) is a Nash equilibrium then, by definition, it prevents exceeding $(x^T R y, x^T C y)$, which implies the following Proposition.

Proposition 1. *If (x, y) is a Nash equilibrium, $v \geq x^T R y$, and $u \geq x^T C y$, then $\text{PE}(v, u)$ has a solution and it prevents exceeding (v, u) .*

By the following proposition, in order to find an ε -well-supported Nash equilibrium it suffices to find a strategy profile that prevents exceeding $(\varepsilon, \varepsilon)$.

Proposition 2. *If a strategy profile (x, y) prevents exceeding (v, u) then it is a $\max(v, u)$ -well-supported Nash equilibrium.*

Proof. Let $i' \in \text{supp}(x)$ and let $i \in \{1, 2, \dots, n\}$. Then we have:

$$R_{i\bullet}y - R_{i'\bullet}y \leq R_{i\bullet}y \leq v,$$

where the first inequality follows from $R_{i'\bullet}y \geq 0$, and the other one holds because y prevents exceeding v . Similarly, and using the assumption that x prevents exceeding u , we can argue that for all $j' \in \text{supp}(y)$ and $j \in \{1, 2, \dots, n\}$, we have $x^T C_{\bullet j} - x^T C_{\bullet j'} \leq u$. It follows that (x, y) is a $\max(v, u)$ -well-supported Nash equilibrium. $\square$

3.2 Strategies that well support a payoff

Definition 6 (Well supporting payoffs). *We say that a strategy $x \in [0, 1]^n$ for the row player well supports $v \in [0, 1]$ against a strategy $y \in [0, 1]^n$ for the column player if for every $i \in \text{supp}(x)$, we have $R_{i\bullet}y \geq v$. Similarly, we say that a strategy $y \in [0, 1]^n$ for the column player well supports $u \in [0, 1]$ against a strategy $x \in [0, 1]^n$ for the row player if for every $j \in \text{supp}(y)$, we have $x^T C_{\bullet j} \geq u$.*

For brevity, we say that a strategy profile (x, y) well supports (v, u) if x well supports v against y and y well supports u against x .

The following theorem states that the payoffs of every Nash equilibrium can be nearly well supported by a strategy profile with supports of constant size.

Theorem 2. *Let (x^*, y^*) be a Nash equilibrium. For every $\delta > 0$, there are $\kappa(\delta)$ -uniform strategies x, y such that the strategy profile (x, y) well supports $((x^*)^T R y^* - \delta, (x^*)^T C y^* - \delta)$, where $\kappa(\delta) = \lceil 2 \ln(1/\delta) / \delta^2 \rceil$.*

The proof of this technical result is postponed until Section 4.

Let $v, u \in [0, 1]$, $\delta > 0$, and let $\mathcal{I}$ and $\mathcal{J}$ be multisets of pure strategies of size $\kappa(\delta)$. Consider the following system $\text{WS}(v, u, \mathcal{I}, \mathcal{J}, \delta)$ of linear constraints:

$$\begin{aligned} x_i &= k_i / \kappa(\delta) \quad \text{for all } i = 1, 2, \dots, n; \\ y_j &= \ell_j / \kappa(\delta) \quad \text{for all } j = 1, 2, \dots, n; \\ R_{i\bullet}y &\geq v - \delta \quad \text{for all } i \in \mathcal{I}; \\ x^T C_{\bullet j} &\geq u - \delta \quad \text{for all } j \in \mathcal{J}; \end{aligned}$$

where k_i is the number of times i occurs in multiset $\mathcal{I}$, and ℓ_j is the number of times j occurs in multiset $\mathcal{J}$. Note that the system $\text{WS}(v, u, \mathcal{I}, \mathcal{J}, \delta)$ of linear constraints characterizes $\kappa(\delta)$ -uniform strategy profiles (x, y) , such that $\text{supp}(x) = \mathcal{I}$ and $\text{supp}(y) = \mathcal{J}$, that well support $(v - \delta, u - \delta)$. Theorem 2 implies the following.

Corollary 1. *If (x, y) is a Nash equilibrium, $v \leq x^T R y$, $u \leq x^T C y$, and $\delta > 0$, then there are multisets $\mathcal{I}$ and $\mathcal{J}$ from $\{1, 2, \dots, n\}$ of size $\kappa(\delta)$, such that $\text{WS}(v, u, \mathcal{I}, \mathcal{J}, \delta)$ has a solution and it well supports $(v - \delta, u - \delta)$.*

By the following proposition, in order to find an ε -well-supported Nash equilibrium it suffices to find a strategy profile that well supports $(1 - \varepsilon, 1 - \varepsilon)$.

Proposition 3. *If a strategy profile (x, y) well supports (v, u) then it is a $(1 - \min(v, u))$ -well-supported Nash equilibrium.*

Proof. Let $i' \in \text{supp}(x)$ and let $i \in \{1, 2, \dots, n\}$. Then we have:

$$R_{i\bullet}y - R_{i'\bullet}y \leq 1 - R_{i'\bullet}y \leq 1 - v,$$

where the first inequality follows from $R_{i\bullet}y \leq 1$, and the other one holds because y well supports v . Similarly, and using the assumption that x well supports u , we can argue that for all $j' \in \text{supp}(y)$ and $j \in \{1, 2, \dots, n\}$, we have $x^T C_{\bullet j} - x^T C_{\bullet j'} \leq 1 - u$. It follows that (x, y) is a $(1 - \min(v, u))$ -well-supported Nash equilibrium. $\square$

3.3 The algorithm for symmetric games

Propositions 2 and 3 suggest that in order to identify a $1/2$ -well-supported Nash equilibrium it suffices to find either a strategy profile that prevents exceeding $(1/2, 1/2)$ or one that well supports $(1/2, 1/2)$. Moreover, verifying existence and identifying such strategy profiles can be done efficiently by solving the linear program $\text{PE}(1/2, 1/2)$, and by solving linear programs $\text{WS}(1/2 + \delta, 1/2 + \delta, \mathcal{I}, \mathcal{J}, \delta)$ for all multisets $\mathcal{I}$ and $\mathcal{J}$ of pure strategies of size $\kappa(\delta)$, respectively.

For arbitrary bimatrix games the above scheme may fail if none of these systems of linear constraints has a solution. Note, however, that—by Proposition 1 and Corollary 1—it would indeed succeed if we could guarantee that the game had a Nash equilibrium with both payoffs at most $1/2$, or with both payoffs at least $(1/2 + \delta)$. Symmetric bimatrix games nearly satisfy this requirement thanks to existence of symmetric Nash equilibria in every symmetric game [12].

If (x^*, x^*) is a symmetric Nash equilibrium in a symmetric bimatrix game (R, R^T) then—trivially—either $(x^*)^T R x^* \leq 1/2$ or $(x^*)^T R x^* > 1/2$. In the former case, by Proposition 1 the linear program $\text{PE}(1/2, 1/2)$ has a solution, and by Proposition 2 it is a $(1/2)$ -well-supported Nash equilibrium. In the latter case, by Corollary 1 there are multisets $\mathcal{I}$ and $\mathcal{J}$ of pure strategies of size $\kappa(\delta)$, such that $\text{WS}(1/2, 1/2, \mathcal{I}, \mathcal{J}, \delta)$ has a solution (x, y) and it well supports $(1/2 - \delta, 1/2 - \delta)$. It then follows by Proposition 3 that (x, y) is a $(1/2 + \delta)$ -well-supported Nash equilibrium.

Algorithm 1 *Let (R, R^T) be a symmetric game and let $\delta > 0$.*

1. *If $\text{PE}(1/2, 1/2)$ has a solution x then return (x, x) .*
2. *Otherwise, that is if $\text{PE}(1/2, 1/2)$ does not have a solution:*

- (a) Using exhaustive search, find multisets $\mathcal{I}$ and $\mathcal{J}$ of pure strategies, both of size $\kappa(\delta)$, such that $WS(1/2, 1/2, \mathcal{I}, \mathcal{J}, \delta)$ has a solution.
(b) Return a solution (x, y) of $WS(1/2, 1/2, \mathcal{I}, \mathcal{J}, \delta)$. $\square$

In order to find appropriate $\mathcal{I}$ and $\mathcal{J}$ in step 2(a), an exhaustive enumeration of all pairs of multisets $\mathcal{I}$ and $\mathcal{J}$ of size $\kappa(\delta)$ is done, and for each such pair the system of linear constraints $WS(1/2, 1/2, \mathcal{I}, \mathcal{J}, \delta)$ is solved. Note that the number of $\kappa(\delta)$ -element multisets from an n -element set is

$$\binom{n + \kappa(\delta) - 1}{\kappa(\delta)} = n^{O(\kappa(\delta))} = n^{O(\ln(1/\delta)/\delta^2)}.$$

Therefore, step 2. of the algorithm requires solving $n^{O(\ln(1/\delta)/\delta^2)}$ linear programs and hence the algorithm runs in time $N^{O(\ln(1/\delta)/\delta^2)}$.

Theorem 3. *For every $\delta > 0$, Algorithm 1 runs in time $N^{O(\ln(1/\delta)/\delta^2)}$ and it returns a strategy profile that is a $(1/2 + \delta)$ -well-supported Nash equilibrium.*

4 Proof of Theorem 2

We use the probabilistic method: random $\kappa(\delta)$ -uniform strategies are drawn by sampling $\kappa(\delta)$ pure strategies (with replacement) from the distributions x^* and y^* , respectively, and Hoeffding's inequality is used to show that the probability of thus selecting a strategy profile that well supports $(v^* - \delta, u^* - \delta)$ is positive if $\kappa(\delta) \geq 2 \ln(1/\delta)/\delta^2$, where $v^* = (x^*)^T R y^*$ and $u^* = (x^*)^T C y^*$.

Consider $2\kappa(\delta)$ mutually independent random variables I_t and J_t , $1 \leq t \leq \kappa(\delta)$, with values in $\{1, 2, \dots, n\}$, the former with the same distribution as strategy x^* and the latter with the same distribution as strategy y^* , that is we have $\mathbb{P}\{I_t = i\} = x_i^*$ and $\mathbb{P}\{J_t = j\} = y_j^*$ for $i, j = 1, 2, \dots, n$. Define the random distributions $X = (X_1, X_2, \dots, X_n)$ and $Y = (Y_1, Y_2, \dots, Y_n)$, with values in $[0, 1]^n$, by setting:

$$X_i = \frac{1}{\kappa(\delta)} \cdot \sum_{t=1}^{\kappa(\delta)} [I_t = i] \quad \text{and} \quad Y_j = \frac{1}{\kappa(\delta)} \cdot \sum_{t=1}^{\kappa(\delta)} [J_t = j].$$

Note that every realization of Y is a $\kappa(\delta)$ -uniform strategy that uses the pure strategy j , $1 \leq j \leq n$, with probability $K_j/\kappa(\delta)$, where $K_j = \sum_{t=1}^{\kappa(\delta)} [J_t = j]$ is the number of indices t , $1 \leq t \leq \kappa(\delta)$, for which $J_t = j$. A similar characterization holds for every realization of X . Observe also that $\text{supp}(X) \subseteq \text{supp}(x^*)$ and $\text{supp}(Y) \subseteq \text{supp}(y^*)$ because for all i and j , $1 \leq i, j \leq n$, the random variables X_i and Y_j are identically equal to 0 unless $x_i^* > 0$ and $y_j^* > 0$, respectively.

Since we want (a realization of) the random strategies X and Y to well support a certain pair of values, we now characterize $R_{i\bullet} Y$, for all $i \in \text{supp}(x^*)$; the whole reasoning presented below for $R_{i\bullet} Y$ can be carried out analogously for $X^T C_{\bullet j}$, for all $j = 1, 2, \dots, n$, and hence it is omitted.

First, observe that for all $i = 1, 2, \dots, n$, we have:

$$R_{i\bullet}Y = \sum_{j=1}^n R_{ij}Y_j = \frac{1}{\kappa(\delta)} \cdot \sum_{j=1}^n R_{ij} \cdot \sum_{t=1}^{\kappa(\delta)} [J_t = j] = \frac{1}{\kappa(\delta)} \cdot \sum_{t=1}^{\kappa(\delta)} R_{iJ_t}.$$

Therefore, the random variable $R_{i\bullet}Y$ is equal to the arithmetic average

$$\overline{Z}_i = \frac{1}{\kappa(\delta)} \cdot \sum_{t=1}^{\kappa(\delta)} Z_{it}$$

of the independent random variables $Z_{it} = R_{iJ_t}$, $1 \leq t \leq \kappa(\delta)$.

For every $i \in \text{supp}(x^*)$, we will apply Hoeffding's inequality to the corresponding random variable $\overline{Z}_i$. Hoeffding's inequality gives an exponential upper bound for the probability of large deviations of the arithmetic average of independent and bounded random variables from their expectation.

Lemma 1 (Hoeffding's inequality). *Let $Z_1, Z_2, \dots, Z_k$ be independent random variables with $0 \leq Z_t \leq 1$ for every t , let $\overline{Z} = (1/k) \cdot \sum_{t=1}^k Z_t$, and let $\mathbb{E}\{\overline{Z}\}$ be its expectation. Then for all $\delta > 0$, we have $\mathbb{P}\{\overline{Z} - \mathbb{E}\{\overline{Z}\} \leq -\delta\} \leq e^{-2\delta^2 k}$.*

Before we apply Hoeffding's inequality to the random variables $\overline{Z}_i$ defined above, observe that for every $t = 1, 2, \dots, \kappa(\delta)$, we have:

$$\mathbb{E}\{Z_{it}\} = \mathbb{E}\{R_{iJ_t}\} = \sum_{j=1}^n R_{ij} \cdot \mathbb{P}\{J_t = j\} = R_{i\bullet}y^*.$$

Note, however, that if $i \in \text{supp}(x^*)$ then $\mathbb{E}\{Z_{it}\} = R_{i\bullet}y^* = v^*$, because (x^*, y^*) is a Nash equilibrium, and hence every $i \in \text{supp}(x^*)$ is a best response to y^* . It follows that $\mathbb{E}\{\overline{Z}_i\} = (1/\kappa(\delta)) \cdot \sum_{t=1}^{\kappa(\delta)} \mathbb{E}\{Z_{it}\} = v^*$.

Applying Hoeffding's inequality, for every $i \in \text{supp}(x^*)$, we get:

$$\mathbb{P}\{R_{i\bullet}Y < v^* - \delta\} = \mathbb{P}\{\overline{Z}_i - \mathbb{E}\{\overline{Z}_i\} < -\delta\} \leq e^{-2\delta^2 \kappa(\delta)}. \quad (1)$$

It follows that if $I \subseteq \text{supp}(x^*)$ and $|I| \leq \kappa(\delta)$, then:

$$\begin{aligned} \mathbb{P}\{R_{i\bullet}Y < v^* - \delta \text{ for some } i \in I\} &\leq \\ &\leq \sum_{i \in I} \mathbb{P}\{R_{i\bullet}Y < v^* - \delta\} \leq \kappa(\delta) \cdot e^{-2\delta^2 \kappa(\delta)} = 2\delta^2 \ln(1/\delta) < \frac{1}{2}, \end{aligned} \quad (2)$$

for all $\delta > 0$. The first inequality holds by the union bound, and the second follows from (1) and because $|I| \leq \kappa(\delta)$. The last inequality can be verified by observing that the function $f(x) = 2x^2 \ln(1/x)$, for $x > 0$, achieves its maximum at $x = 1/\sqrt{e}$ and $f(1/\sqrt{e}) = 1/e < 1/2$.

In a similar way we can prove that if $J \subseteq \text{supp}(y^*)$ and $|J| \leq \kappa(\delta)$, then:

$$\mathbb{P}\{X^T C_{\bullet j} < (x^*)^T C y^* - \delta \text{ for some } j \in J\} < \frac{1}{2}, \quad (3)$$

for all $\delta > 0$.

We are now ready to argue that

$$\mathbb{P}\{R_{i\bullet}Y \geq v^* - \delta \text{ for all } i \in \text{supp}(X), \\ \text{and } X^T C_{\bullet j} \geq u^* - \delta \text{ for all } j \in \text{supp}(Y)\} > 0,$$

and hence there must be realizations $x, y \in [0, 1]^n$ of the random variables $X = (X_1, X_2, \dots, X_n)$ and $Y = (Y_1, Y_2, \dots, Y_n)$, such that (x, y) well supports $(v^* - \delta, u^* - \delta)$. Indeed, we have:

$$\begin{aligned} & \mathbb{P}\{R_{i\bullet}Y < v^* - \delta \text{ for some } i \in \text{supp}(X), \\ & \quad \text{or } X^T C_{\bullet j} < u^* - \delta \text{ for some } j \in \text{supp}(Y)\} \\ & \leq \sum_{I \subseteq \text{supp}(x^*)} \mathbb{P}\{I = \text{supp}(X) \text{ and } R_{i\bullet}Y < v^* - \delta \text{ for some } i \in I\} \\ & \quad + \sum_{J \subseteq \text{supp}(y^*)} \mathbb{P}\{J = \text{supp}(Y) \text{ and } X^T C_{\bullet j} < u^* - \delta \text{ for some } j \in J\} \\ & = \sum_{\substack{I \subseteq \text{supp}(x^*) \\ |I| \leq \kappa(\delta)}} \mathbb{P}\{I = \text{supp}(X)\} \cdot \mathbb{P}\{R_{i\bullet}Y < v^* - \delta \text{ for some } i \in I \mid I = \text{supp}(X)\} \\ & \quad + \sum_{\substack{J \subseteq \text{supp}(y^*) \\ |J| \leq \kappa(\delta)}} \mathbb{P}\{J = \text{supp}(Y)\} \cdot \mathbb{P}\{X^T C_{\bullet j} < u^* - \delta \text{ for some } j \in J \mid J = \text{supp}(Y)\} \\ & < \sum_{I \subseteq \text{supp}(x^*)} \mathbb{P}\{I = \text{supp}(X)\} \cdot \frac{1}{2} + \sum_{J \subseteq \text{supp}(y^*)} \mathbb{P}\{J = \text{supp}(Y)\} \cdot \frac{1}{2} = 1, \end{aligned}$$

where the first inequality follows from the union bound, and from $\text{supp}(X) \subseteq \text{supp}(x^*)$ and $\text{supp}(Y) \subseteq \text{supp}(y^*)$; the equality holds because $|\text{supp}(X)| \leq \kappa(\delta)$ and $|\text{supp}(Y)| \leq \kappa(\delta)$ by the definitions of X and Y ; and the latter (strict) inequality follows from (2) and (3).

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