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Hopf bifurcation analysis for a two-neuron network with four delays ☆

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Abstract

A delay-differential system modelling an artificial neural network with two neurons is investigated. At appropriate parameter values, linear stability and Hopf bifurcation including its direction and stability of the network with four delays are established in this paper. The main tools to obtain our results are the normal form method and the center manifold theory introduced by Hassard. Simulations show that the theoretically predicted values are in excellent agreement with the numerically observed behavior. Our results extend and complement some earlier publications.

Keywords: Hopf bifurcation; Neural network; Delay; Linear stability

1. Introduction

The investigation of neural networks has been the subject of much recent activity, to more or less represent the nervous system, or at least to design systems which can perform tasks associated with higher functions of the human central nervous system [19]. One class of models has come to be as Artificial Neural Networks, since their architecture only metaphorically resemble that of the animal nervous system: pattern recognition problems in particular have a prime target for these investigations. What came to be known as Hopfield Networks has been a prominent tool in the elaboration of these systems.

As pointed out in [25], neural networks are complex and large-scale nonlinear dynamics, while the dynamics of the delayed neural network is even richer and more complicated. To obtain a deep and clear understanding of the dynamics of neural networks, one of the usual ways is to investigate the delayed neural network models with two neurons, see

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[1–10,19,21,22,24,27]. It is hoped that, through discussing the dynamics of two neuron networks, we can get some light for our understanding about the large networks.

In this paper, we shall study a two-neuron network modelled by the following nonlinear differential system

$$\begin{cases} \dot{x}(t) = -x(t) + a_{11}f(x(t-\tau_1)) + a_{12}f(y(t-\tau_2)), \\ \dot{y}(t) = -y(t) + a_{21}f(x(t-\tau_3)) + a_{22}f(y(t-\tau_4)), \end{cases} \tag{1.1}$$

where, $\dot{x}(t) = \mathrm{d}x(t)/\mathrm{d}t,\ x(t)$ and $y(t)$ denote the activations of two neurons, $\tau_i, i = 1,2,3,4$ denote the synaptic transmission delays, $1 \leqslant i, j \leqslant 2$ are the synaptic weights, $f : \mathbb{R} \to \mathbb{R}$ is the activation function. Such a model describes the computational performance of a Hopfield network, where, each neuron is represented by a linear circuit consisting of a resistor and a capacitor, and each neuron is connected to another via the nonlinear activation function f multiplied by the synaptic weights $a_{ij}(i \neq j)$. We allow that a neuron has self-feedback and delayed signal transmission which is due to the finite switching speed of neurons (for more details about delayed neural networks, we refer to [25]).

In general, the stability and bifurcation analysis for such a system is very hard due to the multiple delays. Works on the analysis of such network would face a transcendental equation with more exponential terms. In [17], Mahaffy et al. proved that finding all the parameter values for all the roots of a two order transcendental equation with two delays to have negative real parts is a hopeless task.

As the facts mentioned above, we always assume that $a_{11} = a_{12} = a_{21} = a_{22} = a$, $\tau_1 + \tau_4 = \tau_2 + \tau_3 = 2\tau$ and $f : \mathbb{R} \to \mathbb{R}$ is a C^3-smooth increasing function with $f(0) = 0$ throughout this paper. Note that bifurcations in neural network models and other differential equations with one or two delays have been studied by many researchers [4,6,7,9,13–16,18,26–28]. However, there are few papers discussed the bifurcations of the neural network models with multiple delays (four delays). This fact motivates our work for the paper. We will use the coefficient and the first derivative of the activation function instead of the delay as the bifurcation parameter to get the Hopf bifurcation including its direction and stability, which is different from the previous articles and hence this work is a complement to the previous mentioned one. The main tools to obtain our results are the normal form method and the center manifold theory introduced by Hassard [12].

The outline of this paper is as follows. In Section 2, we shall discuss the associated characteristic equation, the linear stability and Hopf bifurcations; Section 3 is devoted to the direction and stability analysis of the Hopf bifurcation; Numerical simulations are given in Section 4, and we conclude this paper in Section 5.

2. Stability analysis and Hopf bifurcation

Without loss of generality, we may assume that $\tau_1 \geqslant \tau_2 \geqslant \tau_3 \geqslant \tau_4$. Let $C([-\tau_1, 0],\ \mathbb{R}^2)$ denote the Banach space of continuous mapping from $[-\tau_1, 0]$ into $\mathbb{R}^2$ equipped with the supremum norm $\|\phi\| = \sup_{-\tau_1 \leqslant \theta \leqslant 0}|\phi(\theta)|$ for $\phi \in C([-\tau_1, 0],\ \mathbb{R}^2)$. In what follows, if $\sigma \in \mathbb{R}$, $A \geqslant 0$ and $u : [\sigma - \tau_1, \sigma + A] \to \mathbb{R}^2$ is a continuous mapping, then $u_t \in C([-\tau_1, 0],\ \mathbb{R}^2)$, $t \in [\sigma, \sigma + A]$, is defined by $u_t(\theta) = u(t + \theta)$ for $-\tau_1 \leqslant \theta \leqslant 0$.

Linearizing (1.1) at the origin (the trivial solution of (1.1)) leads to

$$\begin{cases} \dot{x}(t) = -x(t) + bx(t-\tau_1) + by(t-\tau_2), \\ \dot{y}(t) = -y(t) + bx(t-\tau_3) + by(t-\tau_4), \end{cases} \tag{2.1}$$

where $b = af'(0)$. We first determine when the infinitesimal generator $A(b)$ of the C^0-semigroup generated by the linear system (2.1) has a pair of purely imaginary eigenvalues. The characteristic equation for this linear DDE is obtained by considering solutions with the form

$$y(t) = \mathrm{e}^{\lambda t}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Such solutions will be nontrivial if and only if the determinant of the following matrix

$$\Delta(\tau, \lambda) = \begin{pmatrix} \lambda + 1 - b\mathrm{e}^{-\lambda\tau_1} & -b\mathrm{e}^{-\lambda\tau_2} \\ -b\mathrm{e}^{-\lambda\tau_3} & \lambda + 1 - b\mathrm{e}^{-\lambda\tau_4} \end{pmatrix} \tag{2.2}$$

is zero, i.e.,

$$\det \Delta(\tau, \lambda) = (\lambda + 1 - b\mathrm{e}^{-\lambda\tau_1})(\lambda + 1 - b\mathrm{e}^{-\lambda\tau_4}) - b^2\mathrm{e}^{-\lambda(\tau_2+\tau_3)} = 0, \tag{2.3}$$

which can be simplified as

$$\det \Delta(\tau, \lambda) = (\lambda + 1)(\lambda + 1 - b\mathrm{e}^{-\lambda\tau_1} - b\mathrm{e}^{-\lambda\tau_4}) = 0. \tag{2.4}$$

It is well known that the trivial solution of the nonlinear DDE (1.1) is locally asymptotically stable if all roots λ of the characteristic Eq. (2.3) satisfy $\mathrm{Re}(\lambda) < 0$ (please refer to [11]). As $\lambda = -1$ is a negative root of Eq. (2.4), therefore, we only consider the following transcendental equation

$$\lambda + 1 - b\mathrm{e}^{-\lambda\tau_1} - b\mathrm{e}^{-\lambda\tau_4} = 0. \tag{2.5}$$

Notice that $\lambda = 0$ is a real root of Eq. (2.5) when $b = \frac{1}{2}$. Meanwhile, we know that $i\beta(\beta > 0)$ is a root of Eq. (2.5) if and only if β satisfies

$$\mathrm{i}\beta + 1 = b[(\cos\beta\tau_1 + \cos\beta\tau_4) - i(\sin\beta\tau_1 + \sin\beta\tau_4)]. \tag{2.6}$$

Separating the real and imaginary parts, we have

$$\begin{cases} b(\cos\beta\tau_1 + \cos\beta\tau_4) = 1, \\ b(\sin\beta\tau_1 + \sin\beta\tau_4) = -\beta, \end{cases} \tag{2.7}$$

which is equivalent to

$$\begin{cases} 2b\cos\beta\tau\cos\frac{\tau_1-\tau_4}{2}\beta = 1, \\ 2b\sin\beta\tau\cos\frac{\tau_1-\tau_4}{2}\beta = -\beta. \end{cases} \tag{2.8}$$

By (2.8), we have

$$\tan\beta\tau = -\beta. \tag{2.9}$$

Now, we built up two functions Γ_1:$y = \tan(\beta\tau)$, Γ_2: $y = -\beta$ defined on the right hand half-plane. Let (β, y) be a point of intersection of Γ_1 and Γ_2, then the first component of (β, y) is a solution to (2.9). We know the Eq. (2.9) has a sequence of roots $\{\beta_j\}_{j\geqslant 1}$, where (see Fig. 1)

$$\beta_j \in \left(\frac{(2j-1)\pi}{2\tau}, \frac{(2j+1)\pi}{2\tau}\right), \quad j \in \mathbb{N}. \tag{2.10}$$

Define

$$b_j = \frac{1}{\cos\beta\tau\cos\frac{\tau_1-\tau_4}{2}\beta}, \; j \in \mathbb{N}. \tag{2.11}$$

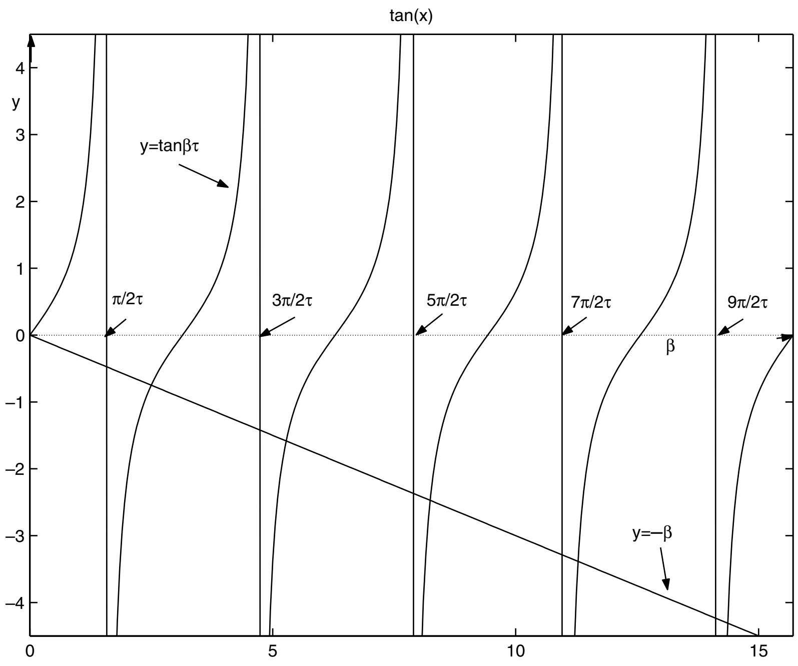

Fig. 1. Numerical solution Γ_1 and Γ_2.

We have the following claim.

Claim. *Eq. (2.4) has purely imaginary roots if and only if* $b = b_j$, *and the purely imaginary roots are* $\pm i\beta_j$, *where* b_j *is defined by (2.11) and* β_j *is a root of (2.9). Furthermore, note the fact that when*

$$\beta \in \left(\frac{2(j-1)\pi}{\tau} + \frac{\pi}{2\tau}, \frac{2(j-1)\pi}{\tau} + \frac{\pi}{\tau}\right), \quad j \in \mathbb{N}, \tag{2.12}$$

we have $\frac{1}{\cos\beta\tau} < 0$, *and when*

$$\beta \in \left(\frac{2(j-1)\pi}{\tau} + \frac{3\pi}{2\tau}, \frac{2(j-1)\pi}{\tau} + \frac{2\pi}{\tau}\right), \quad j \in \mathbb{N}, \tag{2.13}$$

we have we have $\frac{1}{\cos\beta\tau} > 0$, *we can denote* $\{b_j, j \in \mathbb{N}\} = \{b_k^+ | b_k^+ > 0, k \in \mathbb{N}\} \cup \{b_l^- | b_l^- < 0, l \in \mathbb{N}\}$. *The following lemmas play an important role in analyzing the distribution of zeros of* $det\Delta(\tau, \lambda)$.

Lemma 2.1. *Let*

$$E(\lambda) = \lambda + 1 - b e^{-\lambda\tau_1} - b e^{-\lambda\tau_4}, \tag{2.14}$$

then we have the following results:

(*i*) *if* $-\frac{1}{2} < b < \frac{1}{2}$, *then all zeros of* $E(\lambda)$ *have negative real parts*;
(*ii*) *all zeros of* $E(\lambda)$ *have negative real parts when* $b \in (b_1^-, b_1^+)$, *where* $b_1^- = \max_{j\geqslant 1}\{b_k^-\}$, $b_1^+ = \min_{j\geqslant 1}\{b_k^+\}$;
(*iii*) *except for* $\pm i\beta_1^{\mp}$, *all others zeros of* $E(\lambda)$ *have negative real parts when* $b = b_1^{\mp}$;
(*iv*) *if* $|b| > \frac{1}{2}$, *then the following inequality hold*

$$\mathrm{Re}\left[\frac{d\lambda(b)}{db}\bigg|_{b=b_j^+}\right] > 0, \quad \mathrm{Re}\left[\frac{d\lambda(b)}{db}\bigg|_{b=b_j^-}\right] < 0.$$

Proof

(i) Substituting $\lambda = \alpha + i\beta$ into the right side of (2.14) and separating the real and imaginary parts, we obtain

$$R\{E(\alpha,\beta)\} = \alpha + 1 - b(e^{-\alpha\tau_1}\cos(\beta\tau_1) + e^{-\alpha\tau_4}\cos(\beta\tau_4)),$$
$$I\{E(\alpha,\beta)\} = \beta + b(e^{-\alpha\tau_1}\sin(\beta\tau_1) + e^{-\alpha\tau_4}\sin(\beta\tau_4)).$$

Let $R(\alpha) = \alpha + 1 - |b|(e^{-\alpha\tau_1} + e^{-\alpha\tau_4})$. It is obvious that $R\{E(\alpha,\beta)\} \geqslant R(\alpha)$ for all $\alpha \geqslant 0$. In addition, for $-\frac{1}{2} < b < \frac{1}{2}$, we have $R(0) > 0$. This, combined with $\frac{dR(\alpha)}{d\alpha} = 1 + |b|(\tau_1 e^{-\alpha\tau_1} + \tau_4 e^{-\alpha\tau_4}) > 0$, implies that $R(\alpha) > 0$ for $\alpha \geqslant 0$. Therefore,

$$R\{E(\alpha,\beta)\} > 0 \quad \text{for} \quad \alpha \geqslant 0 \quad \text{and} \quad \beta \in \mathbb{R}. \tag{2.15}$$

That is to say if $-\frac{1}{2} < b < \frac{1}{2}$, then all zeros of $E(\lambda)$ have negative real parts.

(ii) Note the fact that b_1^+ is the minimum positive parameter when $E(\lambda)$ has a pair of imaginary roots and b_1^- is the maximum negative parameter when $E(\lambda)$ has a pair of imaginary roots. Using the D-division method of the exponential polynomial introduced by Qin (see p.141, [20]), we can prove that all zeros of $E(\lambda)$ have negative real parts when $b \in (b_1^-, b_1^+)$.

(iii) By way of contradiction, assume that $E(\lambda)$ has a zero with positive real part when $b = b_1^+$. Denote this root as $\lambda(b_0)$, then $\mathrm{Re}\lambda(b_0) > 0$. As the fact that the roots of $E(\lambda)$ are continuous on the exponential polynomial's parameters, we can choose another positive parameter $b_0' < b_0$ such that $\mathrm{Re}\lambda(b_0') > 0$. Which contradicts conclusion (ii). Similarly, we can prove that all roots except for $\pm i\beta_1^-$ have negative real parts when $b = b_1^-$.

(iv) Substituting $\lambda(b)$ into (2.14) and taking the derivative with respect to b, we get

$$\frac{d\lambda(\mu)}{d\mu} = \frac{e^{-\lambda\tau}}{\tau\mu e^{-\lambda\tau} + 1}.$$

Then

$$\frac{d\lambda(b)}{db}\bigg|_{b=b_j^+} = \frac{\lambda + 1}{b_j^+[1 + b_j^+(\tau_1 e^{-\lambda\tau_1} + \tau_4 e^{-\lambda\tau_4})]} = \frac{i\beta_j^+ + 1}{\triangle_1},$$

where

$$\triangle_1 = [b_j^+ + {b_j^+}^2(\tau_1 \cos(\beta_j^+\tau_1) + \tau_4 \cos(\beta_j^+\tau_4))] - \mathrm{i}{b_j^+}^2[\tau_1 \sin(\beta_j^+\tau_1) + \tau_4 \sin(\beta_j^+\tau_4)].$$

Thus

$$\mathrm{Re}\left[\frac{\mathrm{d}\lambda(b)}{\mathrm{d}b}\bigg|_{b=b_j^+}\right] = \frac{\triangle_2}{\triangle_3},$$

where

$$\triangle_2 = b_j^+[1 + b_j^+(\tau_1 \cos(\beta_j^+\tau_1) + \tau_4 \cos(\beta_j^+\tau_4)) - \beta_j^+ b_j^+(\tau_1 \sin(\beta_j^+\tau_1) + \tau_4 \sin(\beta_j^+\tau_4))]$$

and

$$\triangle_3 = {b_j^+}^2[1 + b_j^+(\tau_1 \cos(\beta_j^+\tau_1) + \tau_4 \cos(\beta_j^+\tau_4))]^2 + {b_j^+}^4[\tau_1 \sin(\beta_j^+\tau_1) + \tau_4 \sin(\beta_j^+\tau_4)]^2.$$

Set

$$g_1(\beta_j^+) = \sin(\beta_j^+\tau_1) + \sin(\beta_j^+\tau_4) \quad \text{and} \quad g_2(\beta_j^+) = \cos(\beta_j^+\tau_1) + \cos(\beta_j^+\tau_4).$$

From (2.7)–(2.9), we have $\frac{g_1(\beta_j^+)}{g_2(\beta_j^+)} = \tan\frac{\tau_1+\tau_4}{2}\beta_j^+$. It is easy to get

$$g_1'(\beta_j^+)g_2(\beta_j^+) - g_2'(\beta_j^+)g_1(\beta_j^+) > 0 \quad \text{based on} \quad \tan'(\cdot) > 0.$$

In view of (2.7), we have

$$\triangle_2 = b_j^+[1 + {b_j^+}^2(g_1'(\beta_j^+)g_2(\beta_j^+) - g_2'(\beta_j^+)g_1(\beta_j^+))] > 0. \tag{2.16}$$

Therefore,

$$\mathrm{Re}\left[\frac{\mathrm{d}\lambda(b)}{\mathrm{d}b}\bigg|_{b=b_j^+}\right] > 0.$$

Similarly, we can prove that

$$\mathrm{Re}\left[\frac{\mathrm{d}\lambda(b)}{\mathrm{d}b}\bigg|_{b=b_j^-}\right] < 0.$$

This completes the proof. □

Based on the lemma presented in the above, we have the following results:

Theorem 2.1.

(*i*) *If* $-\frac{1}{2} < b < \frac{1}{2}$, *then all eigenvalues of the generator* $A(b)$ *have negative real parts for all* $\tau_i \geqslant 0$, $i = 1,2,3,4$. *Namely, the equilibrium* (0,0) *of system* (1.1) *is delay-independently locally asymptotically stable*;
(*ii*) *When* $b \in (b_1^-, b_1^+)$, *the equilibrium* (0,0) *of system* (1.1) *is locally asymptotically stable*;
(*iii*) *When* $b \in (-\infty, b_1^-) \cup (b_1^+, +\infty)$, *the equilibrium* (0,0) *of system* (1.1) *is unstable*;
(*iv*) *System* (1.1) *undergoes a Hopf bifurcation at* $b = b_k^{\mp}$, *for* $k = 1,2\ldots$

Proof. (i), (ii) are obvious. From Lemma 2.1 (vi), when $b \in (-\infty, b_1^-) \cup (b_1^+, +\infty)$, $\det\Delta(\tau,\lambda)$ has at least one positive real part zero, this immediately lead to (iii). Claim and Lemma 2.1 (iv) provide the assumptions of the Hopf-theorem (see, p.332 in [11]). This completes the proof. □

Remark 2.1. In this section, we have discussed the associated characteristic equation, the linear stability and Hopf bifurcations of (1.1) with four delays. In fact, our methods can be applied to a broad of system. For example, when $a_{11} = a_{22} = a$ and $a_{12} = a_{21} = -a$, this method is also effective.

3. Direction and stability of the bifurcation

In this section, formulae for determining the direction of Hopf bifurcation and the stability of bifurcating periodic solution of system (2.1) at $b = b_1^- \triangleq \mu$ shall be presented by employing the normal form method and center manifold theorem introduced by Hassard et al. [12]. More precisely, we will compute the reduced system on the center manifold with the pair of conjugate complex, purely imaginary solutions of the characteristic Eq. (2.3). By this reduction we can determine the Hopf bifurcation direction, i.e., to answer the question of whether the bifurcation branch of periodic solution exists locally for supercritical bifurcation or subcritical bifurcation.

Letting $u(t) = (x(t), y(t))^{\mathrm{T}}$ and $u_t(\theta) = u(t+\theta)$ for $\theta \in [-\tau_1, 0]$, we can rewrite Eq. (1.1) as

$$\dot{u}(t) = L_\mu + G(u_t, \mu), \tag{3.1}$$

with

$$L_\mu \varphi = -I\varphi(0) + B_1\varphi(-\tau_1) + B_2\varphi(-\tau_2) + B_3\varphi(-\tau_3) + B_4\varphi(-\tau_4), \tag{3.2}$$

and

$$G(\varphi, \mu) = \frac{f''(0)}{2} a \begin{pmatrix} \varphi_1^2(-\tau_1) + \varphi_2^2(-\tau_2) \\ \varphi_1^2(-\tau_3) + \varphi_2^2(-\tau_4) \end{pmatrix} + \frac{f'''(0)}{6} a \begin{pmatrix} \varphi_1^3(-\tau_1) + \varphi_2^3(-\tau_2) \\ \varphi_1^3(-\tau_3) + \varphi_2^3(-\tau_4) \end{pmatrix} + O(|\varphi|^4), \tag{3.3}$$

where I is the identity matrix, and

$$B_1 = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, B_4 = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}.$$

Then L_μ is a one-parameter family of bounded linear operators in $C([-\tau_1, 0],\ \mathbb{R}^2)$. By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta, \mu)$ in $[-\tau_1, 0] \to \mathbb{R}^4$, such that

$$L_\mu \varphi = \int_{-\tau_1}^0 \mathrm{d}\eta(\theta, \mu)\varphi(\theta) \quad \text{for} \quad \varphi \in C([-\tau_1, 0], \mathbb{R}^2). \tag{3.4}$$

Next, we define for $\varphi \in C^1([-\tau_1, 0],\ \mathbb{R}^2)$,

$$A_\mu \varphi = \begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}\theta}, & \text{for} \quad \theta \in [-\tau_1, 0), \\ \int_{-\tau_1}^0 \mathrm{d}\eta(\xi, \mu)\varphi(\xi) = L_\mu \varphi, & \text{for} \quad \theta = 0 \end{cases} \tag{3.5}$$

and

$$R_\mu \varphi = \begin{cases} 0, & \text{for} \quad \theta \in [-\tau_1, 0), \\ G(\varphi, \mu), & \text{for} \quad \theta = 0. \end{cases} \tag{3.6}$$

Since $\frac{\mathrm{d}u_t}{\mathrm{d}\theta} = \frac{\mathrm{d}u_t}{\mathrm{d}t}$, system (3.1) can be rewritten as

$$\dot{u}_t = A_\mu u_t + R_\mu u_t, \tag{3.7}$$

which is an equation of the form we desired. For $\theta \in [-\tau_1, 0)$, (3.7) is just the trivial equation $\frac{\mathrm{d}u_t}{\mathrm{d}\theta} = \frac{\mathrm{d}u_t}{\mathrm{d}t}$; for $\theta = 0$, it is (3.1).

Denote $A_\mu = A_0$, $R_\mu = R_0$, $L_\mu = L_0$, $\eta(\theta, 0) = \eta(\theta)$. For $\psi \in C^1([0, \tau_1],\ \mathbb{R}^2)$, the adjoint operator A_0^* of A_0 is defined as

$$A_0^*\psi(s) = \begin{cases} -\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & \text{for} \quad s \in (0, \tau_1], \\ \int_{-\tau_1}^0 \mathrm{d}\eta^{\mathrm{T}}(t)\psi(-t), & \text{for} \quad s = 0, \end{cases} \tag{3.8}$$

where η^{T} denotes the transpose of η (recall that L_μ is real). Note that the domains of A_0 and A_0^* are $C^1([-\tau_1, 0]$ and $C^1([0, \tau_1]$, respectively, where for convenience in computation we shall allow functions with range $\mathbb{C}^2$ instead of $\mathbb{R}^2$.

For $\varphi \in C([-\tau_1, 0]$ and $\psi \in C([0, \tau_1]$, define the bilinear form

$$\langle \psi, \varphi \rangle = \bar{\psi}^{\mathrm{T}}(0) \cdot \varphi(0) - \int_{\theta=-\tau_1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}^{\mathrm{T}}(\xi - \theta) \mathrm{d}\eta(\theta) \varphi(\xi) \mathrm{d}\xi. \tag{3.9}$$

Then A_0^* and A_0 are adjoint operators. Let $q(\theta)$ be an eigenvector for A_0 corresponding to μ_0, namely,

$$A_0 q(\theta) = \mathrm{i}\beta_0 q(\theta). \tag{3.10}$$

By discussion in Section 2, we know that $\pm \mathrm{i}\beta_0$ are eigenvalues of A_0 and other eigenvalues have strictly negative real parts. Thus, they are also eigenvalues of A_0^*. Then, we have the following lemma.

Lemma 3.1. *$q(\theta) = V e^{i\beta_0 \theta}$, $\theta \in [-\tau_1, 0]$, is the eigenvector of A_0 corresponding to $i\beta_0$; $q^*(s) = DV^* e^{i\beta_0 s}$, $s \in [0, \tau_1]$, is the eigenvector of A^* corresponding to $-i\beta_0$, and*

$$\langle q^*, q \rangle = 1, \quad \langle q^*, \bar{q} \rangle = 0, \tag{3.11}$$

where

$$V = (1, \rho_1)^T, \quad V^* = (\rho_2, 1)^T,$$
$$\rho_1 = \frac{1 + i\beta_0 - be^{-i\beta_0\tau_1}}{be^{-i\beta_0\tau_2}}, \quad \rho_2 = \frac{1 - i\beta_0 - be^{i\beta_0\tau_1}}{be^{i\beta_0\tau_2}},$$
$$\overline{D} = [\overline{V}^{*^T} V + \tau_1 e^{-i\beta_0\tau_1} \overline{V}^{*^T} B_1 V + \tau_2 e^{-i\beta_0\tau_2} \overline{V}^{*^T} B_2 V$$
$$+ \tau_3 e^{-i\beta_0\tau_3} \overline{V}^{*^T} B_3 V + \tau_4 e^{-i\beta_0\tau_4} \overline{V}^{*^T} B_4 V]^{-1}.$$

Proof. From (3.5), we can rewrite (3.10) as

$$\frac{\mathrm{d}q(\theta)}{\mathrm{d}\theta} = \mathrm{i}\beta_0 q(\theta), \quad \theta \in [-\tau_1, 0), \tag{3.12}$$
$$L_0 q(0) = \mathrm{i}\beta_0 q(0), \quad \theta = 0. \tag{3.13}$$

From (3.12), we can obtain

$$q(\theta) = V \mathrm{e}^{\mathrm{i}\beta_0 \theta}, \quad \theta \in [-\tau_1, 0], \tag{3.14}$$

where $V = (v_1, v_2)^{\mathrm{T}} \in \mathbb{C}^2$ is a constant vector. Base on (3.13) and (3.14), we have

$$[(1 + \mathrm{i}\beta_0)I - B_1 \mathrm{e}^{-\mathrm{i}\beta_0\tau_1} - B_2 \mathrm{e}^{-\mathrm{i}\beta_0\tau_2} - B_3 \mathrm{e}^{-i}\beta_0\tau_3 - B_4 \mathrm{e}^{-\mathrm{i}\beta_0\tau_4}]V = 0.$$

So, we can choose

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1+\mathrm{i}\beta_0 - be^{-\mathrm{i}\beta_0\tau_1}}{be^{-\mathrm{i}\beta_0\tau_2}} \end{pmatrix} = \begin{pmatrix} 1 \\ \rho_1 \end{pmatrix}. \tag{3.15}$$

It follows from (3.10) that $-\mathrm{i}\beta_0$ is an eigenvalue for A_0^*, and

$$A_0^* q^*(\xi) = -\mathrm{i}\beta_0 q^*(\xi), \tag{3.16}$$

for some nonzero row-vector function $q^*(\xi)\, \xi \in [0, \tau_1]$. By some simple computation, we can get

$$\int_{-\tau_1}^{0} \mathrm{d}\eta^{\mathrm{T}}(t)\psi(-t) = -I\psi(0) + B_1^{\mathrm{T}}\psi(\tau_1) + B_2^{\mathrm{T}}\psi(\tau_2) + B_3^{\mathrm{T}}\psi(\tau_3) + B_4^{\mathrm{T}}\psi(\tau_4). \tag{3.17}$$

Let

$$q^*(\theta) = DV^* \mathrm{e}^{\mathrm{i}\beta_0 \xi}, \quad \xi \in [0, \tau_1], \tag{3.18}$$

where $D=(d_1,d_2)^{\mathrm{T}}$, $V^*=(v_1^*,v_2^*)^{\mathrm{T}}\in\mathbb{C}^2$ are constant vectors. Similarly to the proof of ((3.12)–(3.15)), we obtain

$$V^*=\begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix}=\begin{pmatrix} \frac{1-\mathrm{i}\beta_0-b\mathrm{e}^{\mathrm{i}\beta_0\tau_1}}{b\mathrm{e}^{\mathrm{i}\beta_0\tau_2}} \\ 1 \end{pmatrix}=\begin{pmatrix} \rho_2 \\ 1 \end{pmatrix}. \tag{3.19}$$

Now, we compute $\langle q^*,q\rangle$ as follows

$$\begin{aligned}\langle q^*,q\rangle &=\bar{q}^{*^{\mathrm{T}}}(0)q(0)-\int_{-\tau_1}^{0}\int_{\xi=0}^{\theta}\bar{q}^{*^{\mathrm{T}}}(\xi-\theta)[\mathrm{d}\eta(\theta)]q(\xi)\mathrm{d}\xi \\ &=\overline{D}\left[\overline{V}^{*^{\mathrm{T}}}V-\int_{-\tau_1}^{0}\int_{\xi=0}^{\theta}\overline{V}^{*^{\mathrm{T}}}\mathrm{e}^{-\mathrm{i}\beta_0(\xi-\theta)}[\mathrm{d}\eta(\theta)]V\mathrm{e}^{\mathrm{i}\beta_0\xi}\mathrm{d}\xi\right]=\overline{D}\left[\overline{V}^{*^{\mathrm{T}}}V-\int_{-\tau_1}^{0}\overline{V}^{*^{\mathrm{T}}}[\mathrm{d}\eta(\theta)]\theta\mathrm{e}^{\mathrm{i}\beta_0\theta}V\right] \\ &=\overline{D}[\overline{V}^{*^{\mathrm{T}}}V+\tau_1\mathrm{e}^{-\mathrm{i}\beta_0\tau_1}\overline{V}^{*^{\mathrm{T}}}B_1V+\tau_2\mathrm{e}^{-\mathrm{i}\beta_0\tau_2}\overline{V}^{*^{\mathrm{T}}}B_2V+\tau_3\mathrm{e}^{-\mathrm{i}\beta_0\tau_3}\overline{V}^{*^{\mathrm{T}}}B_3V+\tau_4\mathrm{e}^{-\mathrm{i}\beta_0\tau_4}\overline{V}^{*^{\mathrm{T}}}B_4V].\end{aligned} \tag{3.20}$$

So, when

$$\overline{D}=[\overline{V}^{*^{\mathrm{T}}}V+\tau_1\mathrm{e}^{-\mathrm{i}\beta_0\tau_1}\overline{V}^{*^{\mathrm{T}}}B_1V+\tau_2\mathrm{e}^{-\mathrm{i}\beta_0\tau_2}\overline{V}^{*^{\mathrm{T}}}B_2V+\tau_3\mathrm{e}^{-\mathrm{i}\beta_0\tau_3}\overline{V}^{*^{\mathrm{T}}}B_3V+\tau_4\mathrm{e}^{-\mathrm{i}\beta_0\tau_4}\overline{V}^{*^{\mathrm{T}}}B_4V]^{-1},$$

from (3.20) we can get $\langle q^*,q\rangle=1$. On the other hand, from

$$-\mathrm{i}\beta_0\langle q^*,\bar{q}\rangle=\langle q^*,A_0\bar{q}\rangle=\langle A_0^*q^*,\bar{q}\rangle=\langle -\mathrm{i}\beta_0q^*,\bar{q}\rangle=\mathrm{i}\beta_0\langle q^*,\bar{q}\rangle,$$

therefore, $\langle q^*,\bar{q}\rangle=0$. This completes the proof. □

In the remainder of this section, we use the same notations as in Hassard et al. (see [12]). We first compute the coordinates to describe the center manifold C_0 at $\mu=\mu_0$. Let u_t be the solution of Eq. (3.7) when $\mu=\mu_0$. Define

$$z(t)=\langle q^*,u_t\rangle, \tag{3.21}$$
$$W(t,\theta)=u_t(\theta)-2\mathrm{Re}\{z(t)q(\theta)\}. \tag{3.22}$$

On the center manifold C_0, we have

$$W(t,\theta)=W(z(t),\bar{z}(t),\theta), \tag{3.23}$$

where

$$W(z,\bar{z},\theta)=W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\bar{z}+W_{02}(\theta)\frac{z^2\bar{z}}{2}+W_{30}(\theta)\frac{z^3}{6}+\cdots. \tag{3.24}$$

In fact, z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Note that W is real if u_t is real. We consider only real solutions in this paper.

From (3.22), we get

$$\langle q^*,W\rangle=\langle q^*,u_t(\theta)-2\mathrm{Re}\{z(t)q(\theta)\}\rangle=\langle q^*,u_t(\theta)-z(t)q(\theta)-\bar{z}(t)\bar{q}(\theta)\rangle=\langle q^*,u_t\rangle-z(t)\langle q^*,q\rangle-\bar{z}(t)\langle q^*,\bar{q}\rangle=0.$$

For solution $u_t\in C_0$ of Eq. (3.7), from (3.6) and (3.9), we have

$$\dot{z}(t)=\langle q^*,\dot{u}_t\rangle=\langle q^*,A_0u_t+R_0u_t\rangle=\langle A_0^*q^*,u_t\rangle+\bar{q}^{*^{\mathrm{T}}}(0)G(u_t,0)=\mathrm{i}\beta_0z(t)+\bar{q}^{*^{\mathrm{T}}}(0)f_0(z,\bar{z}), \tag{3.25}$$

which we can write in abbreviated form as

$$\dot{z}(t)=\mathrm{i}\beta_0z(t)+g(z,\bar{z}), \tag{3.26}$$

where

$$g(z,\bar{z})=\bar{q}^{*^{\mathrm{T}}}(0)f_0(z,\bar{z})=\bar{q}^{*^{\mathrm{T}}}(0)G(W(z,\bar{z},\theta)+2\mathrm{Re}\{zq(\theta)\},0)=g_{20}\frac{z^2}{2}+g_{11}z\bar{z}+g_{02}\frac{\bar{z}^2}{2}+g_{21}(\theta)\frac{z^2\bar{z}}{2}+\cdots. \tag{3.27}$$

By (3.7) and (3.26), we have

$$\begin{aligned}\dot{W} &= \dot{u}_t - \dot{z}q - \dot{\bar{z}}\cdot\bar{q} = A_0u_t + R_0u_t - \mathrm{i}\beta_0 zq - \bar{q}^{*^{\mathrm{T}}}(0)f_0(z,\bar{z})q + \mathrm{i}\beta_0\overline{zq} - q^{*^{\mathrm{T}}}(0)\overline{f_0(z,\bar{z})}\bar{q}\\ &= A_0u_t + R_0u_t - A_0(zq) - A_0(\bar{z}\cdot\bar{q}) - 2\mathrm{Re}\{\bar{q}^{*^{\mathrm{T}}}(0)f_0(z,\bar{z})q\} = A_0W + R_0u_t - 2\mathrm{Re}\{\bar{q}^{*^{\mathrm{T}}}(0)f_0(z,\bar{z})q\}\\ &= \begin{cases} A_0W - 2\mathrm{Re}\{q^{*^{\mathrm{T}}}(0)f_0(z,\bar{z})q\}, & \theta\in[-\tau_1,0),\\ A_0W - 2\mathrm{Re}\{q^{*^{\mathrm{T}}}(0)f_0(z,\bar{z})q\} + f_0(z,\bar{z}), & \theta = 0.\end{cases}\end{aligned}$$

We rewrite this as

$$\dot{W} = A_0W + H(z,\bar{z},\theta), \tag{3.28}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + H_{30}(\theta)\frac{z^3}{6} + \cdots$$

and

$$\dot{W} = W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}.$$

Expanding the above series and comparing the coefficients, we obtain

$$\begin{cases}(A_0 - 2\mathrm{i}\beta_0)W_{20}(\theta) = -H_{20}(\theta),\\ A_0W_{11}(\theta) = -H_{11}(\theta),\\ (A_0 + 2\mathrm{i}\beta_0)W_{02}(\theta) = -H_{02}(\theta),\\ \cdots\cdots\end{cases} \tag{3.29}$$

Notice that

$$\begin{aligned}x(t-\tau_1) &= W^{(1)}(t,-\tau_1) + z(t)\mathrm{e}^{-\mathrm{i}\beta_0\tau_1} + \bar{z}(t)\mathrm{e}^{\mathrm{i}\beta_0\tau_1}\\ y(t-\tau_2) &= W^{(2)}(t,-\tau_2) + z(t)N + \bar{z}(t)\overline{N}\\ x(t-\tau_3) &= W^{(1)}(t,-\tau_3) + z(t)\mathrm{e}^{-\mathrm{i}\beta_0\tau_3} + \bar{z}(t)\mathrm{e}^{\mathrm{i}\beta_0\tau_3}\\ y(t-\tau_4) &= W^{(2)}(t,-\tau_4) + z(t)N\mathrm{e}^{-\mathrm{i}\beta_0(\tau_4-\tau_2)} + \bar{z}(t)\overline{N}\mathrm{e}^{\mathrm{i}\beta_0(\tau_4-\tau_2)},\end{aligned}$$

where

$$\begin{aligned}N &= \frac{1}{b}(1 + \mathrm{i}\beta_0 - b\mathrm{e}^{-\mathrm{i}\beta_0\tau_1})\\ W^{(1)}(t,-\tau_1) &= W_{20}^{(1)}(-\tau_1)\frac{z^2(t)}{2} + W_{11}^{(1)}(-\tau_1)z(t)\bar{z}(t) + W_{02}^{(1)}(-\tau_1)\frac{\bar{z}^2(t)}{2} + \cdots\\ W^{(2)}(t,-\tau_2) &= W_{20}^{(2)}(-\tau_2)\frac{z^2(t)}{2} + W_{11}^{(2)}(-\tau_2)z(t)\bar{z}(t) + W_{02}^{(2)}(-\tau_2)\frac{\bar{z}^2(t)}{2} + \cdots\\ W^{(1)}(t,-\tau_3) &= W_{20}^{(1)}(-\tau_3)\frac{z^2(t)}{2} + W_{11}^{(1)}(-\tau_3)z(t)\bar{z}(t) + W_{02}^{(1)}(-\tau_3)\frac{\bar{z}^2(t)}{2} + \cdots\\ W^{(2)}(t,-\tau_4) &= W_{20}^{(j)}(-\tau_4)\frac{z^2(t)}{2} + W_{11}^{(j)}(-\tau_4)z(t)\bar{z}(t) + W_{02}^{(j)}(-\tau_4)\frac{\bar{z}^2(t)}{2} + \cdots.\end{aligned}$$

It follows that

$$
\begin{aligned}
g(z,\bar z) &= \bar q^{*^{\mathrm T}}(0) f_0(z,\bar z)\\
&= \bar q^{*^{\mathrm T}}(0) G(W(z,\bar z,\theta)+2\mathrm{Re}\{zq(\theta)\},0)\\
&= \frac{f''(0)}{2} a\overline{D}(M,1)\begin{pmatrix} x^2(t-\tau_1)+y^2(t-\tau_2)\\ x^2(t-\tau_3)+y^2(t-\tau_4)\end{pmatrix}\\
&\quad + \frac{f'''(0)}{6} a\overline{D}(M,1)\begin{pmatrix} x^3(t-\tau_1)+y^3(t-\tau_2)\\ x^3(t-\tau_3)+y^3(t-\tau_4)\end{pmatrix}+\cdots\\
&= \frac{1}{2}\overline{D}aM\left[f''(0)x^2(t-\tau_1)+\frac{f'''(0)}{3}x^3(t-\tau_1)\right]\\
&\quad + \frac{1}{2}\overline{D}aM\left[f''(0)y^2(t-\tau_2)+\frac{f'''(0)}{3}y^3(t-\tau_2)\right]\\
&\quad + \frac{1}{2}\overline{D}a\left[f''(0)x^2(t-\tau_3)+\frac{f'''(0)}{3}x^3(t-\tau_3)\right]\\
&\quad + \frac{1}{2}\overline{D}a\left[f''(0)y^2(t-\tau_4)+\frac{f'''(0)}{3}y^3(t-\tau_4)\right]+\cdots\\
&= \frac{1}{2}\overline{D}aMf''(0)[z^2\mathrm{e}^{-2\mathrm{i}\beta_0\tau_1}+\bar z^2\mathrm{e}^{2\mathrm{i}\beta_0\tau_1}+2z\bar z\\
&\quad + W_{20}^{(1)}(t,-\tau_1)z^2\bar z\mathrm{e}^{\mathrm{i}\beta_0\tau_1}+2W_{11}^{(1)}(t,-\tau_1)z^2\bar z\mathrm{e}^{-\mathrm{i}\beta_0\tau_1}]\\
&\quad + \frac{1}{2}\overline{D}aMf''(0)[z^2N^2+\bar z^2\overline{N}^2+2z\bar zN\overline{N}+ W_{20}^{(2)}(t,-\tau_2)z^2\bar z\overline{N}+2W_{11}^{(2)}(t,-\tau_2)z^2\bar zN]\\
&\quad + \frac{1}{2}\overline{D}af''(0)[z^2\mathrm{e}^{-2\mathrm{i}\beta_0\tau_3}+\bar z^2\mathrm{e}^{2\mathrm{i}\beta_0\tau_3}+2z\bar z+W_{20}^{(1)}(t,-\tau_3)z^2\bar z\mathrm{e}^{\mathrm{i}\beta_0\tau_3}+2W_{11}^{(1)}(t,-\tau_3)z^2\bar z\mathrm{e}^{-\mathrm{i}\beta_0\tau_3}]\\
&\quad + \frac{1}{2}\overline{D}af''(0)[z^2N^2\mathrm{e}^{-2\mathrm{i}\beta_0(\tau_4-\tau_2)}+\bar z^2\overline{N}^2\mathrm{e}^{2\mathrm{i}\beta_0(\tau_4-\tau_2)}+2z\bar zN\overline{N}\\
&\quad + W_{20}^{(2)}(t,-\tau_4)z^2\bar z\overline{N}\mathrm{e}^{\mathrm{i}\beta_0(\tau_4-\tau_2)}+2W_{11}^{(2)}(t,-\tau_4)z^2\bar zN\mathrm{e}^{-\mathrm{i}\beta_0(\tau_4-\tau_2)}]\\
&\quad + \frac{1}{2}\overline{D}aMf'''(0)[z^2\bar z\mathrm{e}^{-\mathrm{i}\beta_0\tau_1}]+\frac{1}{2}\overline{D}aMf'''(0)[z^2\bar zN^2\overline{N}]\\
&\quad + \frac{1}{2}\overline{D}af'''(0)[z^2\bar z\mathrm{e}^{-\mathrm{i}\beta_0\tau_3}]+\frac{1}{2}\overline{D}af'''(0)[z^2\bar zN^2\overline{N}\mathrm{e}^{-\mathrm{i}\beta_0(\tau_4-\tau_2)}]+\cdots,
\end{aligned}
\tag{3.30}
$$

where

$$M=\frac{1+\mathrm{i}\beta_0-b\mathrm{e}^{-\mathrm{i}\beta_0\tau_1}}{b\mathrm{e}^{-\mathrm{i}\beta_0\tau_2}}=\overline{\rho_2}.$$

Comparing the coefficients with (3.27), we have

$$
\begin{aligned}
g_{20} &= \overline{D}af''(0)[M\mathrm{e}^{-2\mathrm{i}\beta_0\tau_1}+MN^2+\mathrm{e}^{-2\mathrm{i}\beta_0\tau_3}+N^2\mathrm{e}^{-2\mathrm{i}\beta_0(\tau_4-\tau_2)}]\\
g_{11} &= \overline{D}af''(0)[M+MN\overline{N}+1+N\overline{N}]\\
g_{02} &= \overline{D}af''(0)[M\mathrm{e}^{2\mathrm{i}\beta_0\tau_1}+M\overline{N}^2+\mathrm{e}^{2\mathrm{i}\beta_0\tau_3}+\overline{N}^2\mathrm{e}^{2\mathrm{i}\beta_0(\tau_4-\tau_2)}]\\
g_{21} &= \overline{D}af''(0)[MW_{20}^{(1)}(t,-\tau_1)\mathrm{e}^{\mathrm{i}\beta_0\tau_1}+MW_{20}^{(2)}(t,-\tau_2)\overline{N}\\
&\quad + W_{20}^{(1)}(t,-\tau_3)\mathrm{e}^{\mathrm{i}\beta_0\tau_3}+W_{20}^{(2)}(t,-\tau_4)\overline{N}\mathrm{e}^{\mathrm{i}\beta_0(\tau_4-\tau_2)}]\\
&\quad + \overline{D}af'''(0)[M\mathrm{e}^{-\mathrm{i}\beta_0\tau_1}+MN^2\overline{N}+\mathrm{e}^{-\mathrm{i}\beta_0\tau_3}+N^2\overline{N}\mathrm{e}^{-\mathrm{i}\beta_0(\tau_4-\tau_2)}]\\
&\quad + 2\overline{D}af''(0)[MW_{11}^{(1)}(t,-\tau_1)\mathrm{e}^{-\mathrm{i}\beta_0\tau_1}+MW_{11}^{(2)}(t,-\tau_2)N\\
&\quad + W_{11}^{(1)}(t,-\tau_3)N\mathrm{e}^{-\mathrm{i}\beta_0(\tau_3)}+W_{11}^{(2)}(t,-\tau_4)N\mathrm{e}^{-\mathrm{i}\beta_0(\tau_4-\tau_2)}].
\end{aligned}
\tag{3.31}
$$

We still need to compute $W_{11}(t,\theta)$ and $W_{20}(t,\theta)$ for $\theta\in[-\tau_1,0)$. Indeed, we have

$$
\begin{aligned}
H(z,\bar z,\theta) &= -2\mathrm{Re}\{q^{*^{\mathrm T}}(0)f_0(z,\bar z)q(\theta)\} = -2\mathrm{Re}\{g(z,\bar z)q(\theta)\} = -g(z,\bar z)q(\theta)-\bar g(z,\bar z)\bar q(\theta)\\
&= -\left(g_{20}\frac{z^2}{2}+g_{11}z\bar z+g_{02}\frac{\bar z^2}{2}+g_{21}\frac{z^2\bar z}{2}+\cdots\right)q(\theta)-\left(\bar g_{20}\frac{\bar z^2}{2}+\bar g_{11}z\bar z+\bar g_{02}\frac{z^2}{2}+\cdots\right)\bar q(\theta).
\end{aligned}
\tag{3.32}
$$

Comparing the coefficients with (3.28) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta)$$
$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{3.33}$$

It follows from (3.5) and (3.29) that

$$\begin{aligned}\dot{W}_{20}(\theta) &= A_0 W_{20} = 2\mathrm{i}\beta_0 W_{20}(\theta) - H_{20}(\theta) = 2\mathrm{i}\beta_0 W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta)\\ &= 2\mathrm{i}\beta_0 W_{20}(\theta) + g_{20}q(0)\mathrm{e}^{\mathrm{i}\beta_0\theta} + \bar{g}_{02}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\beta_0\theta}.\end{aligned}$$

Solving for $W_{20}(\theta)$, we obtain

$$W_{20}(\theta) = \frac{ig_{20}}{\beta_0}q(0)\mathrm{e}^{\mathrm{i}\beta_0\theta} + \frac{i\bar{g}_{02}}{3\beta_0}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\beta_0\theta} + E_1\mathrm{e}^{2\mathrm{i}\beta_0\theta}, \tag{3.34}$$

and similarly, we obtain

$$W_{11}(\theta) = -\frac{ig_{11}}{\beta_0}q(0)\mathrm{e}^{\mathrm{i}\beta_0\theta} + \frac{i\bar{g}_{11}}{\beta_0}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\beta_0\theta} + E_2,$$

where E_1 and E_2 are both 2-dimensional vectors, and can be determined by setting $\theta = 0$ in $H(z,\bar{z},\theta)$. As $W_{20}(\theta)$ and $W_{11}(\theta)$ are continuous on $[-\tau_1, 0]$, so we have

$$W_{20}(0) = \frac{ig_{20}}{\beta_0}q(0) + \frac{i\bar{g}_{02}}{3\beta_0}\bar{q}(0) + E_1, \tag{3.35}$$

and

$$W_{11}(0) = -\frac{ig_{11}}{\beta_0}q(0) + \frac{i\bar{g}_{11}}{\beta_0}\bar{q}(0) + E_2. \tag{3.36}$$

From

$$\begin{aligned}H(z,\bar{z},0) &= -2\mathrm{Re}\{q^{*^{\mathrm{T}}}(0)f_0(z,\bar{z})q(0)\} + f_0(z,\bar{z})\\ &= -\left(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots\right)q(\theta) - \left(\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \cdots\right)\bar{q}(\theta) + f_0(z,\bar{z}),\end{aligned} \tag{3.37}$$

noting that

$$\begin{aligned}f_0(z,\bar{z}) = G(u_t(\theta),0) &= \frac{f''(0)}{2}a\begin{pmatrix}x^2(-\tau_1)+y^2(-\tau_2)\\ x^2(-\tau_3)+y^2(-\tau_4)\end{pmatrix} + \frac{f'''(0)}{6}a\begin{pmatrix}x^3(-\tau_1)+y^3(-\tau_2)\\ x^3(-\tau_3)+y^3(-\tau_4)\end{pmatrix} + O(|\varphi|^4)\\ &= \frac{f''(0)}{2}\begin{pmatrix}a[W^{(1)}(t,-\tau_1)+z(t)\mathrm{e}^{-\mathrm{i}\beta_0\tau_1}+\bar{z}(t)\mathrm{e}^{\mathrm{i}\beta_0\tau_1}]^2\\ a[W^{(1)}(t,-\tau_3)+z(t)\mathrm{e}^{-\mathrm{i}\beta_0\tau_3}+\bar{z}(t)\mathrm{e}^{\mathrm{i}\beta_0\tau_3}]^2\end{pmatrix}\\ &\quad + \frac{f''(0)}{2}\begin{pmatrix}a[W^{(2)}(t,-\tau_2)+z(t)N+\bar{z}(t)\overline{N}]^2\\ a[W^{(2)}(t,-\tau_4)+z(t)N\mathrm{e}^{-\mathrm{i}\beta_0(\tau_4-\tau_2)}+\bar{z}(t)\overline{N}\mathrm{e}^{\mathrm{i}\beta_0(\tau_4-\tau_2)}]^2\end{pmatrix} + \cdots,\end{aligned} \tag{3.38}$$

and comparing with (3.28), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + f''(0)\begin{pmatrix}a\mathrm{e}^{-2\mathrm{i}\beta_0\tau_1} + aN^2\\ a\mathrm{e}^{-2\mathrm{i}\beta_0\tau_3} + aN^2\mathrm{e}^{-2\mathrm{i}\beta_0(\tau_2-\tau_4)}\end{pmatrix}, \tag{3.39}$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + f''(0)\begin{pmatrix}a + aN\overline{N}\\ a + aN\overline{N}\end{pmatrix}. \tag{3.40}$$

From (3.5) and (3.29), we have

$$-W_{20}(0) + B_1W_{20}(-\tau_1) + B_2W_{20}(-\tau_2) + B_3W_{20}(-\tau_3) + B_4W_{20}(-\tau_4) = 2\mathrm{i}\beta_0W_{20}(0) - H_{20}(0), \tag{3.41}$$

and

$$-W_{11}(0) + B_1W_{11}(-\tau_1) + B_2W_{11}(-\tau_2) + B_3W_{11}(-\tau_3) + B_4W_{11}(-\tau_4) = -H_{11}(0). \tag{3.42}$$

Substituting (3.34), (3.35) into (3.41), we have

$$\frac{ig_{20}}{\beta_0}[(1+\mathrm{i}\beta_0)I - B_1\mathrm{e}^{-\mathrm{i}\beta_0\tau_1} - B_2\mathrm{e}^{-\mathrm{i}\beta_0\tau_2} - B_3\mathrm{e}^{-\mathrm{i}\beta_0\tau_3} - B_4\mathrm{e}^{-\mathrm{i}\beta_0\tau_4}] + \frac{i\overline{g_{02}}}{3\beta_0}[(1-\mathrm{i}\beta_0)I - B_1\mathrm{e}^{\mathrm{i}\beta_0\tau_1} - B_2\mathrm{e}^{\mathrm{i}\beta_0\tau_2} - B_3\mathrm{e}^{\mathrm{i}\beta_0\tau_3} - B_4\mathrm{e}^{\mathrm{i}\beta_0\tau_4}] + [(1+2\mathrm{i}\beta_0)I - B_1\mathrm{e}^{-2\mathrm{i}\beta_0\tau_1} - B_2\mathrm{e}^{-2\mathrm{i}\beta_0\tau_2} - B_3\mathrm{e}^{-2\mathrm{i}\beta_0\tau_3} - B_4\mathrm{e}^{-2\mathrm{i}\beta_0\tau_4}]E_1 = f''(0)\begin{pmatrix} a\mathrm{e}^{-2\mathrm{i}\beta_0\tau_1} + aN^2 \\ a\mathrm{e}^{-2\mathrm{i}\beta_0\tau_3} + aN^2\mathrm{e}^{-2\mathrm{i}\beta_0(\tau_4-\tau_2)} \end{pmatrix}. \tag{3.43}$$

Noting that

$$\begin{aligned}
&[(1+\mathrm{i}\beta_0)I - B_1\mathrm{e}^{-\mathrm{i}\beta_0\tau_1} - B_2\mathrm{e}^{-\mathrm{i}\beta_0\tau_2} - B_3\mathrm{e}^{-\mathrm{i}\beta_0\tau_3} - B_4\mathrm{e}^{-\mathrm{i}\beta_0\tau_4}]q(0)\\
&\quad = \Delta(\tau,\mathrm{i}\beta_0)q(0) = 0,\\
&[(1-\mathrm{i}\beta_0)I - B_1\mathrm{e}^{\mathrm{i}\beta_0\tau_1} - B_2\mathrm{e}^{\mathrm{i}\beta_0\tau_2} - B_3\mathrm{e}^{\mathrm{i}\beta_0\tau_3} - B_4\mathrm{e}^{\mathrm{i}\beta_0\tau_4}]\overline{q(0)}\\
&\quad = \Delta(\tau,-\mathrm{i}\beta_0)\overline{q(0)} = 0,\\
&[(1+2\mathrm{i}\beta_0)I - B_1\mathrm{e}^{-2\mathrm{i}\beta_0\tau_1} - B_2\mathrm{e}^{-2\mathrm{i}\beta_0\tau_2} - B_3\mathrm{e}^{-2\mathrm{i}\beta_0\tau_3} - B_4\mathrm{e}^{-2\mathrm{i}\beta_0\tau_4}]\\
&\quad = \Delta(\tau,2\mathrm{i}\beta_0),
\end{aligned}$$

we get

$$\Delta(\tau,2\mathrm{i}\beta_0) = f''(0)\begin{pmatrix} a\mathrm{e}^{-2\mathrm{i}\beta_0\tau_1} + aN^2 \\ a\mathrm{e}^{-2\mathrm{i}\beta_0\tau_3} + aN^2\mathrm{e}^{-2\mathrm{i}\beta_0(\tau_4-\tau_2)} \end{pmatrix}.$$

Solving the above set of equations for $(E_1^{(1)}, E_1^{(2)})^{\mathrm{T}} = E_1$, we obtain

$$E_1 = f''(0)\Delta^{-1}(\tau,2\mathrm{i}\beta_0)\begin{pmatrix} a\mathrm{e}^{-2\mathrm{i}\beta_0\tau_1} + aN^2 \\ a\mathrm{e}^{-2\mathrm{i}\beta_0\tau_3} + aN^2\mathrm{e}^{-2\mathrm{i}\beta_0(\tau_4-\tau_2)} \end{pmatrix}.$$

Similarly, we have

$$E_2 = f''(0)\Delta^{-1}(\tau,0)\begin{pmatrix} a + aN\overline{N} \\ a + aN\overline{N} \end{pmatrix}.$$

Based on the above analysis, we can see that each g_{ij} in (3.31) is determined by the parameters in system (1.1). Thus, we can compute the following values

$$\begin{aligned}
C_1(0) &= \frac{i}{2\beta_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2},\\
U_2 &= -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\lambda'(\mu_0)},\\
T_2 &= -\frac{\mathrm{Im}\{C_1(0)\} + \mu_2\mathrm{Im}\{\lambda'(\mu_0)\}}{\beta_0},\\
B_2 &= 2\mathrm{Re}\{C_1(0)\}.
\end{aligned} \tag{3.44}$$

which determine the quantities of bifurcating periodic solutions in the center manifold at the critical value μ_0, i.e., U_2 determines the directions of the Hopf bifurcation: if $U_2 > 0$ ($U_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exists for $\mu = \mu_0$; B_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $B_2 < 0$ ($B_2 > 0$); and T_2 determines the period of the bifurcating periodic solutions: the period increase (decrease) if $T_2 > 0$ ($T_2 < 0$). Since $\mathrm{Re}\lambda'(\mu_0) < 0$, we thus have the following result.

Theorem 3.1. *Let* $C_1(0)$ *be given in* (3.44)*. Then*

(*i*) *the bifurcating periodic solutions exists for* $\mu = \mu_0$ *and if*

$$\mathrm{Re}\{C_1(0)\} > 0 (\mathrm{Re}\{C_1(0)\} < 0),$$

then the Hopf bifurcation is supercritical (*subcritical*);

(*ii*) *the bifurcating periodic solutions are stable* (*unstable*) *if*

$\mathrm{Re}\{C_1(0)\} < 0(\mathrm{Re}\{C_1(0)\} > 0)$;

(*iii*) T_2 *determines the period of the bifurcating periodic solutions*: *the period increases* (*decreases*) *if* $T_2 > 0(T_2 < 0)$.

Base on the fact that many network adopt the form of activation function as $f(x) = \tanh x$, *we give more precise conclusion as following.*

Corollary 3.1. *Suppose that* $f'(0) \neq 0, f''(0) = 0, f'''(0) \neq 0$. *Let*

$$m = 2 + \tau_1 + \tau_3 + (\tau_1 + \tau_3)\beta_0^2 + b_0(\tau_2 + \tau_4 - \tau_1 - \tau_3)(\cos\,\beta_0\tau_4 - \beta_0 \sin \beta_0\tau_4),$$

then at $b = b_0$, *system* (1.1) *undergoes a Hopf bifurcation. The direction of Hopf bifurcation and stability of bifurcation periodic solutions are determined by* $\mathrm{sign}\left\{m\frac{f'''(0)}{f'(0)}\right\}$. *More precisely, if* $\mathrm{sign}\left\{m\frac{f'''(0)}{f'(0)}\right\} < 0(>0)$, *then the Hopf bifurcation is subcritical* (*supercritical*) *and the bifurcating periodic solutions are orbitally asymptotically stable* (*unstable*).

Proof. Since $f''(0) = 0$, it follows from (3.31) that $g_{20} = g_{11} = g_{02} = 0$ and

$$g_{21} = \overline{D}f'''(0)a[M\mathrm{e}^{-\mathrm{i}\beta_0\tau_1} + MN^2\overline{N} + \mathrm{e}^{-\mathrm{i}\beta_0\tau_3} + N^2\overline{N}\mathrm{e}^{-\mathrm{i}\beta_0(\tau_4-\tau_2)}]. \tag{3.45}$$

From Lemma 3.1, direct computing, we get $\overline{D} = \frac{1}{H}$, where $H = 2N\mathrm{e}^{\mathrm{i}\beta_0\tau_2} + \tau_1 b_0 N\mathrm{e}^{-\mathrm{i}\beta_0(\tau_1-\tau_2)} + \tau_2 b_0 N^2\mathrm{e}^{\mathrm{i}\beta_0\tau_2} + \tau_3 b_0\mathrm{e}^{-\mathrm{i}\beta_0\tau_3} + \tau_4 b_0 N\mathrm{e}^{-\mathrm{i}\beta_0(\tau_4-\tau_2)}$. Base on the fact that $i\beta_0$ is a root of Eq. (2.5), we have

$$b_0\mathrm{e}^{\mathrm{i}\beta_0(\tau_4-\tau_1)} = (1 + \mathrm{i}\beta_0)\mathrm{e}^{\mathrm{i}\beta_0\tau_4} - b_0. \tag{3.46}$$

From (2.4) and the definition of N, M, it is easy to find that

$$M = N\mathrm{e}^{\mathrm{i}\beta_0\tau_2}, \quad N = \mathrm{e}^{-\mathrm{i}\beta_0\tau_4}. \tag{3.47}$$

Substituting (3.46) and (3.47) into (3.45), we obtain

$$\begin{aligned} C_1(0) &= \frac{g_{21}}{2} = \frac{\overline{D}f'''(0)b_0[M\mathrm{e}^{-\mathrm{i}\beta_0\tau_1} + MN^2\overline{N} + \mathrm{e}^{-\mathrm{i}\beta_0\tau_3} + N^2\overline{N}\mathrm{e}^{-\mathrm{i}\beta_0(\tau_4-\tau_2)}]}{2f'(0)} \\ &= \frac{f'''(0)b_0[N\mathrm{e}^{\mathrm{i}\beta_0(\tau_2-\tau_1)} + N^3\overline{N}\mathrm{e}^{\mathrm{i}\beta_0\tau_2} + \mathrm{e}^{-\mathrm{i}\beta_0\tau_3} + N^2\overline{N}\mathrm{e}^{-\mathrm{i}\beta_0(\tau_4-\tau_2)}]}{2Hf'(0)} \\ &= \frac{f'''(0)b_0[\mathrm{e}^{-\mathrm{i}\beta_0\tau_3} + \mathrm{e}^{-\mathrm{i}\beta_0(2\tau_4-\tau_2)}]}{f'(0)[2\mathrm{e}^{-\mathrm{i}\beta_0(\tau_4-\tau_2)} + \tau_1 b_0\mathrm{e}^{-\mathrm{i}\beta_0\tau_3} + \tau_2 b_0\mathrm{e}^{-\mathrm{i}\beta_0(2\tau_4-\tau_2)} + \tau_3 b_0\mathrm{e}^{-\mathrm{i}\beta_0\tau_3} + \tau_4 b_0\mathrm{e}^{-\mathrm{i}\beta_0(2\tau_4-\tau_2)}]} \\ &= \frac{f'''(0)b_0[\mathrm{e}^{\mathrm{i}\beta_0(\tau_4-\tau_1)} + 1]}{f'(0)[2\mathrm{e}^{\mathrm{i}\beta_0\tau_4} + \tau_1\mathrm{e}^{\mathrm{i}\beta_0(\tau_4-\tau_1)} + \tau_2 b_0 + \tau_3 b_0\mathrm{e}^{\mathrm{i}\beta_0(\tau_4-\tau_1)} + \tau_4 b_0]} \\ &= \frac{f'''(0)(1+\mathrm{i}\beta_0)\mathrm{e}^{\mathrm{i}\beta_0\tau_4}}{f'(0)[2\mathrm{e}^{\mathrm{i}\beta_0\tau_4} + (\tau_1+\tau_3)(1+\mathrm{i}\beta_0)\mathrm{e}^{\mathrm{i}\beta_0\tau_4} + b_0(\tau_2+\tau_4-\tau_1-\tau_3)]} = \frac{f'''(0)}{f'(0)} \times \frac{(1+\mathrm{i}\beta_0)(H_1 - H_2 i)}{H_1^2 + H_2^2}, \end{aligned} \tag{3.48}$$

where, $H_1 = 2 + \tau_1 + \tau_3 + b_0(\tau_2 + \tau_4 - \tau_1 - \tau_3)\quad \cos\beta_0\tau_4$ and $H_2 = (\tau_1 + \tau_3)\beta_0 - b_0(\tau_2 + \tau_4 - \tau_1 - \tau_3)\quad \sin\,\beta_0\tau_4$. Therefore,

$$\mathrm{sign}\{\mathrm{Re}\{C_1(0)\}\} = \mathrm{sign}\left\{\mathrm{Re}\left\{\frac{mf'''(0)}{(H_1^2+H_2^2)f'(0)}\right\}\right\} = \mathrm{sign}\left\{\mathrm{Re}\left\{m\frac{f'''(0)}{f'(0)}\right\}\right\}. \tag{3.49}$$

By Theorem 3.1, we can see the Corollary 3.1 is true. This completes the proof. □

Remark 3.1. To the best of our knowledge, few authors considered the bifurcation for model (1.1) with four delays. We can find [6,8,19], in the existing work. Especially, the authors of [6] suppose that only one delay appears in the system, which is a special case of our results. And the authors of [8] also consider system (1.1) under special conditions: $a_{11} = a_{22}$ and $\tau_1 = \tau_4 = \tau, \tau_2 + \tau_3 = 2\tau$. In addition, we also notice that two conditions in [19] are presented as $a_{11} = a_{22} = 0$, $\tau_1 = \tau_3, \tau_2 = \tau_4$ and $\tau_1 = \tau_2 = \tau_3 = \tau_4$. Obviously, our model is more complex for the transcendental equation with more exponential terms. Furthermore, the methods applied in mentioned papers cannot be applied to our model. Therefore, our conclusions extend and implement these publications.

4. Numerical simulation example

In this section, some numerical results of simulating system (1.1) are presented at different data of b. Using the method of numerical simulation in [23], we will find that the theoretically predicted values are in excellent agreement with the numerically observed behavior.

Example. Consider the system as follows

$$\begin{cases} \dot{x}(t) = -x(t) + b\tanh\left(x\left(t - \frac{13}{12}\pi\right)\right) + b\tanh\left(y\left(t - \frac{11}{12}\pi\right)\right), \\ \dot{y}(t) = -y(t) + b\tanh\left(x\left(t - \frac{7}{12}\pi\right)\right) + b\tanh\left(y\left(t - \frac{5}{12}\pi\right)\right). \end{cases} \tag{4.1}$$

If $b = -0.455$, then $b \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. It follows from Theorem 2.1(i), the equilibrium (0,0) of system (4.1) is delay-independently locally asymptotically stable. Again, a quick computation revealed that $b = -\sqrt{2}$ is the critical value for Hopf bifurcation. Using Corollary 3.1 and direct computation, we obtain $m = 2 + \frac{10}{3}\pi - \frac{\sqrt{2}}{3}\pi\left(\cos\frac{5}{12}\pi - \sin\frac{5}{12}\pi\right) > 0$. Therefore, $\mathrm{Re}\{C_1(0)\} < 0$. That is to say the Hopf bifurcation of system (4.1) is subcritical and the bifurcating periodic solutions are orbitally asymptotically stable when $b = -\sqrt{2}$. If we let $b = 0.07 - \sqrt{2}$, then $b \in (b_1^-, b_1+)$, using Theorem 2.1(ii), the origin is asymptotically stable. These conclusions are verified by the numerical simulations in Figs. 2–7, where $(x(\mu);\ y(\mu)) = (0{:}001;\ 0{:}004)$ for $\mu \in [-13\pi/12; 0]$.

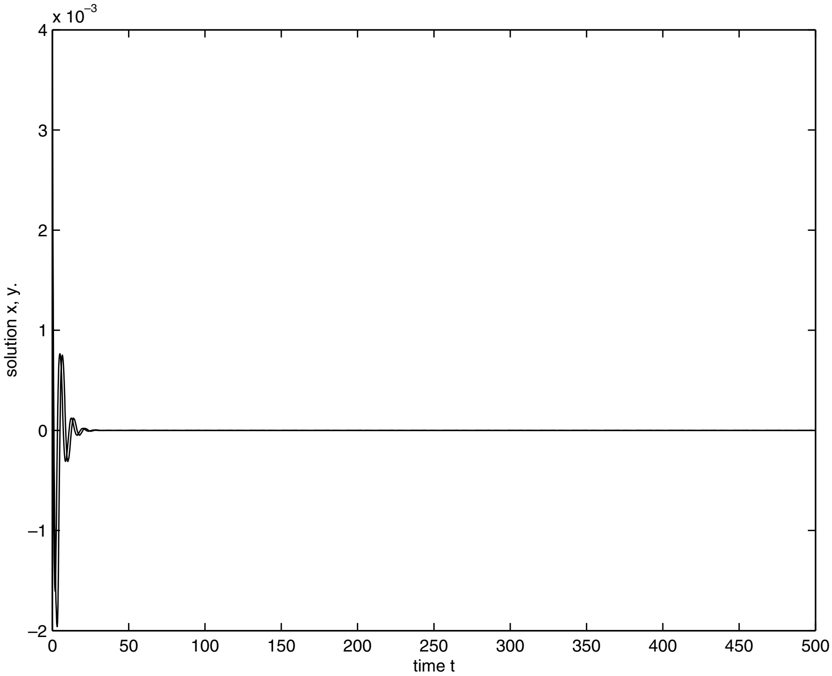

Fig. 2. (0,0) is delay-independently locally asymptotically stable.

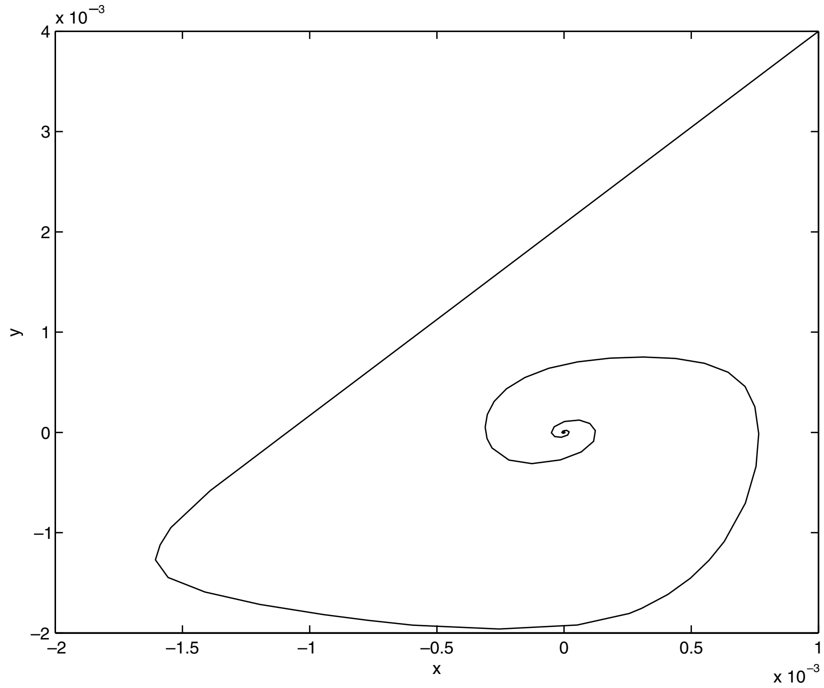

Fig. 3. (0,0) is delay-independently locally asymptotically stable.

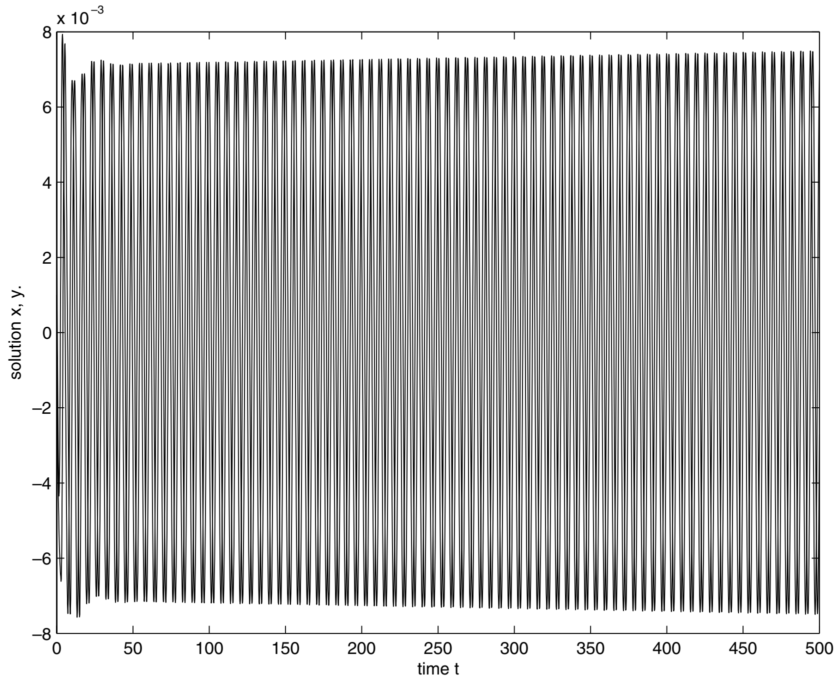

Fig. 4. Periodic solution bifurcates from equilibrium (0,0), where $(x(\theta), y(\theta)) = (0.001, 0.004)$ for $\theta \in \left(-\frac{13\pi}{12}, 0\right]$.

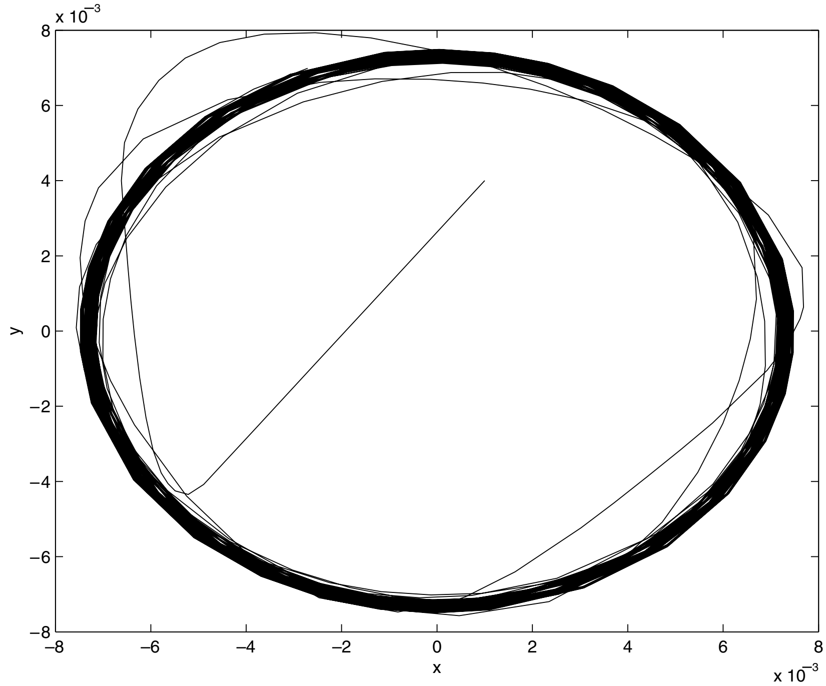

Fig. 5. Periodic solution bifurcates from equilibrium (0,0), where $(x(\theta), y(\theta)) = (0.001, 0.004)$ for $\theta \in \left(-\frac{13\pi}{12}, 0\right]$.

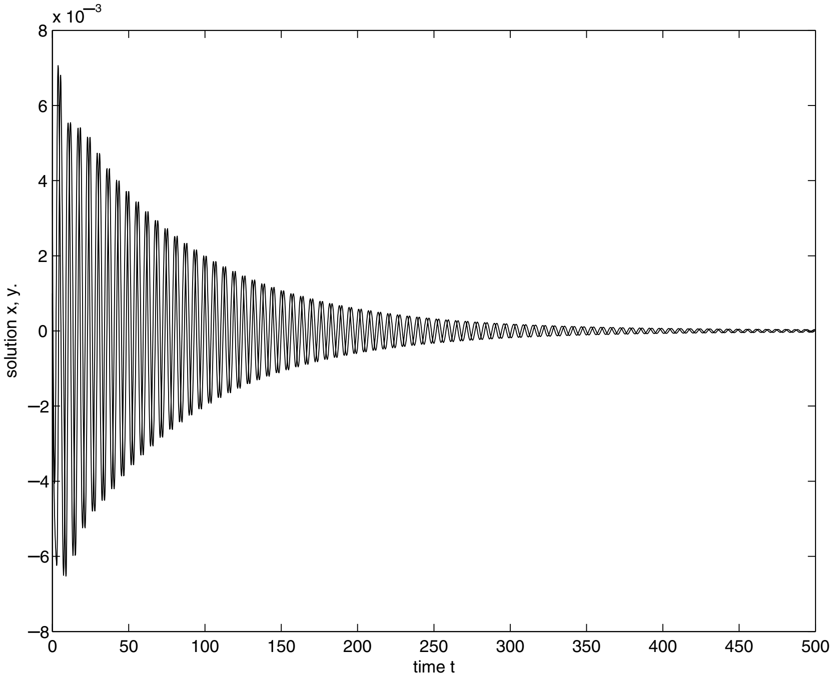

Fig. 6. The equilibrium (0,0) is asymptotically stable.

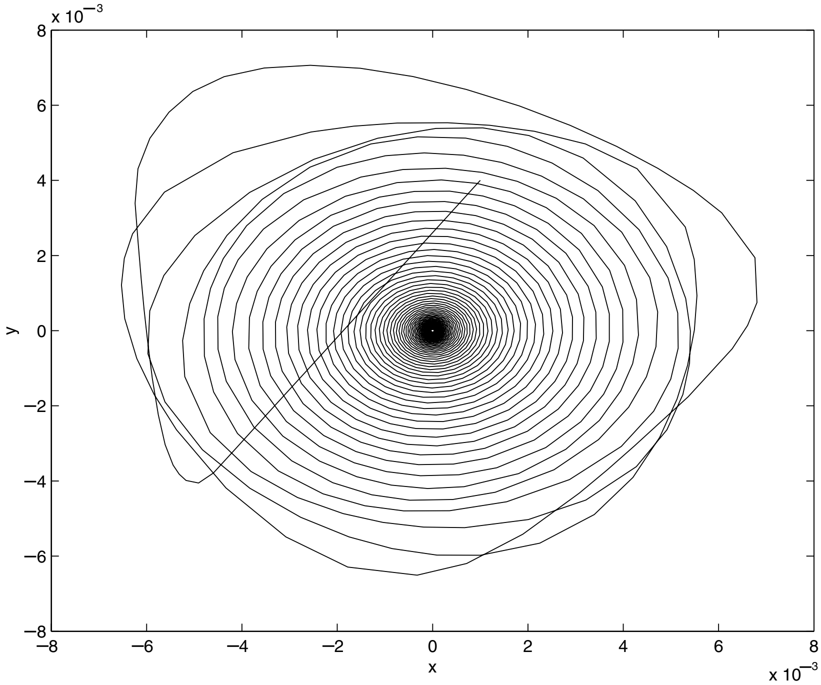

Fig. 7. The equilibrium (0,0) is asymptotically stable.

5. Conclusions

Due to its complexity, the local and Hopf bifurcation analysis for two neuron-network with four delays is far from complete. Just as pointed out by Olien and Bélair [19], it is difficult to find all parameters for all the characteristic roots to have negative real parts. We have derived some sufficient conditions to ensure all the characteristic roots have negative real parts. Using the coefficient number and the first derivative of the activation function as the bifurcation parameter, we also show that a Hopf Bifurcation will occur once this parameter passes through a critical value; i.e., a family of periodic orbits bifurcates from the origin. At last, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying the normal form theory and the center manifold theorem.

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