DCSP-8: Signal Representation
Restoring Memory

- MIMO Biomimetic device mimics signal processing function of hippocampal neural circuits
- Interface with the hippocampus using multi-electrode arrays
- Transform upstream hippocampal signals into downstream hippocampal signals using a computational model
- By-pass damaged hippocampal region
- Restore episodic memory functions
Stochastic process

\[ g(k) = E(x[n + k] - Ex[n + k]) (x[n] - Ex[n]) \]
\[ s(x[n + k]) s(x[n]) \]

- two signals are generated: y (red) is simply \( \text{randn}(1,200) \)
  x (blue) is generated \( x[i+10] = 0.8x[i] + y[i+10] \)

- For y, we have \( \gamma(0)=1, \gamma(n)=0, \) if n is not 0: having no memory
- For x, we have \( \gamma(0)=1, \) and \( \gamma(n) \) is not zero, for some n: having memory
• **White noise** is a random process we can not predict at all (independent of history)

\[
\gamma(k) = \frac{E(x[n+k] - Ex[n+k])(x[n] - Ex[n])}{\sigma(x[n+k])\sigma(x[n])} = 0, \text{ if } k \neq 0
\]

• In other words, it is the most ‘violent’ noise

• White noise draws its name from white light which will become clear in the next few lectures
white noise $w[n]$

- The most ‘noisy’ noise is a white noise since its autocorrelation is zero, i.e.
  \[ \text{corr}(w[n], w[m])=0 \text{ when } n \neq m \]

- Otherwise, we called it colour noise since we can predict some outcome of $w[n]$, given $w[m]$, $m<n$
Why do we love Gaussian?

Sweety Gaussian
Sweety Gaussian

A linear combination of two Gaussian random variables is Gaussian again.

For example, given two independent Gaussian variable $X$ and $Y$ with mean zero,

$aX+bY$ is a Gaussian variable with mean zero and variance $a^2 \sigma(X) + b^2 \sigma(Y)$

This is very rare (the only one in continuous distribution) but extremely useful: panda in the family of all distributions.
Spectrogram: real life example
Sequences and their representation

- A sequence is an *infinite* series of real numbers \( \{ x[n] \} \)

\[
\{ x[n] \} = \{ \ldots, x[-1], x[0], x[1], x[2], \ldots, x[n], \ldots \}
\]

- Represent a sampled signal, i.e. \( x[n] = x(nT) \), where \( x(t) \) is continuous function of time
Sequences and their representation

- A sequence is an *infinite* series of real numbers \( \{ x[n] \} \)

\[
\{ x[n] \} = \{..., x[-1], x[0], x[1], x[2], \ldots, x[n], \ldots \}
\]
The basic tool of signal analysis is

the Fourier transform or DTFT
Definition and properties:

DTFT gives the frequency representation of a discrete time sequence with infinite length.

\[ X(F) = FT(x(t)) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi Ft) dt \]

\[ = \sum_{n=-\infty}^{\infty} x[n] \exp(-j2\pi Fn) \]
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Define ( \( \omega = 2\pi F \) )

\[ X(\omega) = DTFT(x) = x[n]\exp(-j\omega n) \]

Note that \( X(\omega) \) has a period of \( 2\pi \)
Definition and properties:

• Visualize $X(\omega)$

• Plot it in the magnitude and phase plane

DTFT magnitude and phase (right).
The DTFT gives the frequency representation of a discrete time sequence with infinite length.

\[ X(\omega) = DTFT(x) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \]

\[ x[n] = IDTFT(X) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \]

\[ \text{X(\omega): frequency domain} \quad \leftrightarrow \quad x[n]: \text{time domain} \]
Example I

Consider the delta (impulse) function $x[n] = \delta[n]$, we have

$$X( ) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\omega} = e^{0} = 1$$

$x[n] = \delta[n]$, the unit impulse
Example 1

Consider the delta (impulse) function $x[n]=\delta[n]$, we have

$$X(\ ) = [n] \exp(-jn) = \exp(0) = 1_{n=\cdots}$$

amplitude

-\pi \quad \pi

phase

-\pi \quad \pi
Consider the delta (impulse) function $x[n] = \delta[n]$, we have

$$X(\ ) = \sum_{n} [n] \exp(-jn) = \exp(0) = 1$$
Consider the signal

\[ x[n] = 0.5^n, \quad n=0,1,2,\ldots \]

Then

\[
X(w) = \sum_{n=0}^{\infty} 0.5^n \exp(-jn) = \frac{1}{1 - 0.5 \exp(-jw)}
\]

\[
= \frac{1}{1 - 0.5 \cos(w) + j0.5 \sin(w)}
\]

\[
= \frac{1}{1 - 0.5 \cos(w)} \frac{0.5 \sin(w)}{(1 - 0.5 \cos(w))^2 + (0.5 \sin(w))^2} j
\]
Example II

Consider the signal

\[ x[n] = 0.5^n, \quad n = 0, 1, 2, \ldots \]

Then

\[ X(w) = \sum_{n=0}^{\infty} x[n] e^{-jnw} \]

\[ = \sum_{n=0}^{\infty} 0.5^n e^{-jnw} = \frac{1}{1 - 0.5 e^{-jw}} \]

\[ = \frac{1}{1 - 0.5 \cos(w) + j0.5 \sin(w)} = \frac{1}{1 - 0.5 \cos(w) - j0.5 \sin(w)} \]

\[ (1 - 0.5 \cos(w))^2 + (0.5 \sin(w))^2 = 1 - 0.5 \cos(w) \]

\[ (1 - 0.5 \cos(w))^2 + (0.5 \sin(w))^2 - 0.5 \sin(w) \]

\[ \frac{0.5 \sin(w)}{(1 - 0.5 \cos(w))^2 + (0.5 \sin(w))^2} j \]
Consider the signal
\[ x[n] = 0.5^n, \quad n = 0, 1, 2, \ldots \]

Then
\[
X(w) = \frac{1}{0.5^n \exp(jn)} = \frac{1}{1 - 0.5 \exp(-jw)} = \frac{1}{1 - 0.5 \cos(w) + j0.5 \sin(w)} = \frac{1 - 0.5 \cos(w) - j0.5 \sin(w)}{(1 - 0.5 \cos(w))^2 + (0.5 \sin(w))^2} j
\]
Example II

\[ h = 0.01; \]
\[ N = 4000; \]
\[ \text{for } i = 1:N \]
\[ t(i) = (i - N/2) \times h; \]
\[ x(i) = 1/(1 - 0.5 \times \exp(-j \times t(i))); \]
\[ \text{end} \]
\[ \text{figure}(1) \]
\[ \text{plot}(t, \text{abs}(x)); \]
\[ \text{figure}(2) \]
\[ \text{plot}(t, \text{phase}(x)); \]

\begin{align*}
\text{amplitude} & \quad \text{phase} \\
-2 & \quad -0.5 \\
1.2 & \quad 0.5 \\
2 & \quad 0
\end{align*}
Computation of the DTFT

- Fourier approach to signal analysis is based on the expansion of a signal in terms of sinusoids or, more precisely, complex exponentials.

- Analyzing a signal by determining the frequencies contributing to its spectrum in terms of magnitudes and phases.
Computation of the DTFT

• In practice, we do not have infinite memory, we do not have infinite time, and signal does not have infinite duration.

• In addition, the spectrum of the signal changes with time, just as music is composed of different notes that change with time.
Consequently, the DTFT generally is not computable unless we have an analytical expression for the signal we analyze, in which case we can compute it symbolically.
But most of the time this is not the case, especially when we want to determine the frequency spectrum of a signal measured from an experiment.

In this situation, develop an algorithm that can be computed numerically in a finite number of steps such as the discrete Fourier transform (DFT), a special case of DTFT.
DFT and DTFT

• For a given data set
  \[ \{x[1], \ldots x[N]\}, \]
  it is equivalent to
  \[ \{\ldots, 0, x[1], \ldots, x[N], 0, 0,\ldots\}. \]

• Using DTFT, we have

\[
X(\omega) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\omega n)
\]

\[
= \sum_{n=0}^{N-1} x[n + 1] \exp(-j\omega n)
\]
DFT and DTFT

- It turns out that we can sample it (computable with computer),
  at \( \omega = \frac{2\pi k}{N}, \) \( k = 0,1,\ldots,N-1 \)

\[
X(k) = X(2\pi k / N) = \sum_{n=0}^{N-1} x[n + 1] \exp\left(-j \frac{2\pi k}{N} n\right)
\]
Example 1

Consider the delta (impulse) function $x[n] = \delta[n]$, we have

$$X(\cdot) = [n] \exp(jn) = \exp(0) = 1$$
Example I

Sample it (computable with computer), at \( \omega = \frac{2\pi k}{N} \), \( k = 0, 1, \ldots, N-1 \)
Example 1

sample it (computable with computer), at $\omega = 2\pi \frac{k}{N}$, $k = 0, 1, \ldots, N-1$

For example, we have $N = 4$

$2\pi \frac{0}{N} = 0$
$2\pi \frac{1}{N} = \pi/2$
$2\pi \frac{2}{N} = \pi$
$2\pi \frac{3}{N} = 3\pi/4$
Example 1

Sample it (computable with computer), at \( \omega = 2\pi k / N, k = 0,1,\ldots,N-1 \)

For example, we have \( N = 4 \)

\[
2\pi \frac{0}{N} = 0 \\
2\pi \frac{1}{N} = \frac{\pi}{2} \\
2\pi \frac{2}{N} = \pi \\
2\pi \frac{3}{N} = \frac{3\pi}{4}
\]

\[(X[0] \ X[1] \ X[2] \ X[3]) = (1 \ 1 \ 1 \ 1) = DTFT((x[0] \ x[1] \ x[2] \ x[3])) = DFT((1 \ 0 \ 0 \ 0))\]