# On the existence of algebraically natural proofs

Joint work with Prerona Chatterjee, C. Ramya, Ramprasad Saptharishi and Anamay Tengse

# Polynomials

- Main protagonists: multivariate polynomials over a field $\mathbb{F}$
- $P \in \mathbb{F}[x_1, x_2, \ldots, x_n]$, $\deg(P) = d$
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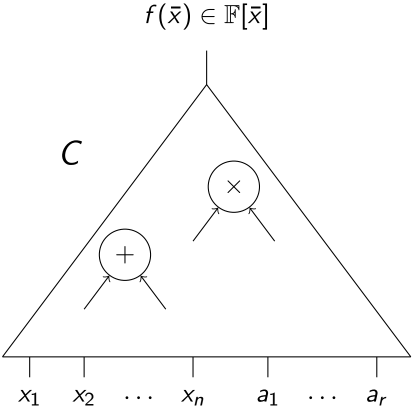
- Variables - $\bar{x} = \{x_1, \ldots, x_n\}$, constants - $\mathbb{F} = \mathbb{C}$
  Operations - Addition $+$ and multiplication $\times$.

# Algebraic Circuits

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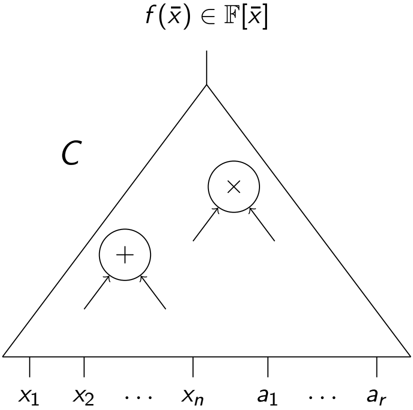


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VP **vs** VNP: Lower bounds for explicit polynomials.

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Can proofs based on this template yield strong lower bounds ?

# The Template

# The template: a toy case

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# Waring rank: a real world example

**Theorem**

*If* $x_1 \cdots x_n = L_1^n + L_2^n + \cdots + L_s^n$ *for linear forms* $L_1, L_2, \ldots, L_s$*, then* $s$ *is at least* $\exp(\Omega(n))$.

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So, for the monomial to be in $\mathcal{C}$, we must have $sn \geq \exp(\Omega(n))$.

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The partial derivative matrix: rows and columns indexed by monomials
$(\alpha, \beta)$ entry $=$ coefficient of the monomial $\beta$ in the partial derivative $\frac{\partial P}{\partial \alpha}$

- Every entry is linear in the coefficients of $P$
- Dim of matrix: $N \times N$ for $N = \binom{n+d}{d}$
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In particular: the determinant of this minor vanishes on coefficient vector of every polynomial in $\mathcal{C}$ and is non-zero on the coefficient vector of $x_1 x_2 \cdots x_n$.

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**Q.** Can we hope to prove superpolynomial lower bounds for algebraic circuits via natural proofs ?

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**Q.** Does VP have an efficiently constructible equations ?[AD,G,FSV,GKSS]

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- Only need to fool algebraic circuits.
- Not enough evidence, one way or the other.

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# Our results

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Restriction **not on circuits** computing the polynomials.

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# Sketch of the Proofs

# Hitting sets for VP

**Definition** (Hitting Set)

$\mathcal{H} \subset \mathbb{F}^n$ is a *hitting set* for a class $\mathcal{C}$ of $n$-variate polynomials, if for all $0 \neq f \in \mathcal{C}$, there exists an $h \in \mathcal{H}$ such that $f(h) \neq 0$.

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$\mathcal{H} \subset \mathbb{F}^n$ is a *hitting set* for a class $\mathcal{C}$ of $n$-variate polynomials, if for all $0 \neq f \in \mathcal{C}$, there exists an $h \in \mathcal{H}$ such that $f(h) \neq 0$.

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Moreover, there is a hitting set with small integer points.

**Observation**: For a nonzero $g$, $g(\mathcal{H}) = 0$ is a proof that $g \notin \mathcal{C}$.

# From hitting set to equations

$\mathcal{H} = \{h_1, \ldots, h_r\}$ hitting set for $\mathcal{C}$, $\qquad 0 \neq g(\bar{x})$ input polynomial.

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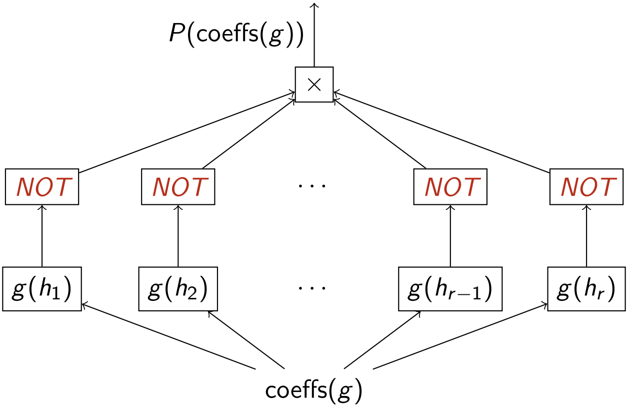


$NOT(0) =$nonzero $\qquad NOT(\text{nonzero}) = 0$

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- Linear polynomial in $\text{coeffs}(g)$.
- We can "hardwire" $\text{eval}(h)$ in our circuit, for all $h \in \mathcal{H}$.

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$$P(\text{coeffs}(g)) \approx \prod_{h \in \mathcal{H}} \left( (\langle \text{coeffs}(g), \text{eval}(h) \rangle)^{q-1} - 1 \right)$$

$\text{Degree}(P) \leq |\mathcal{H}| q \leq \text{poly}(N)$, $\text{Size}(P) \leq \text{poly}(N)$.

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Cannot directly work with eval$(h)$.

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**Chinese Remainder Theorem**

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Set $\ell = \log(LNk^d) = \text{poly}(d, \log N)$. For primes $p_1, \ldots, p_\ell$,
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**Note**: Can "hardwire" $\mathrm{eval}_i(h)$ for all $i \in [\ell]$ and $h \in \mathcal{H}$.

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$$g(h) \neq 0 \iff \exists i \in [\ell] \quad s.t \quad (p_i \nmid \langle \mathsf{coeffs}(g), \mathsf{eval}_i(h) \rangle)$$

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Equation for $\text{VP}'_{\mathbb{C}}$

$$P(\text{coeffs}(g)) \approx \prod_{h \in \mathcal{H}} \prod_{i \in [\ell]} \prod_{\substack{-B \le a \le B \\ p_i \nmid a}} (\langle \text{coeffs}(g), \text{eval}_i(h) \rangle - a)$$

$\text{Deg}(P) \le |\mathcal{H}| \, \text{poly}(n) \, \text{poly}(N) \le \text{poly}(N)$

$\text{Size}(P) \le \text{poly}(N)$.

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This ensures non-triviality of the equations obtained earlier.

# Results for VP

**Theorem** (Equations for $\mathrm{VP}'_{\mathbb{C}}$)

For $n$,$d$ and $N = \binom{n+d}{n}$,

There exists a nonzero $P(Z_1, \dots, Z_N) \in \mathrm{VP}(N)$ such that

for all $f \in \mathrm{VP}_{\mathbb{C}}(n, d)$ with coefficients in $\{-N, \dots, N\}$,

$P(\text{coeffs}(f)) = 0$.

Moreover, there is a $g$ with small coefficients such that
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# Results for VNP

**Theorem** (Equations for $\mathsf{VNP}'_{\mathbb{C}}$)

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There exists a nonzero $Q(Z_1, \ldots, Z_N) \in \mathsf{VP}(N)$ such that
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$Q(\text{coeffs}(f)) = 0$.

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- Efficiently constructible equations exist for polynomials with "small" coefficients, in both VP and VNP.

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- How about seemingly simpler models...formulas/constant depth circuits?
- Limitations on what can be proved via algebraically natural proofs ?

# Thanks!