Simple and Fast Derandomization from Very Hard Functions

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In this talk

› the setting

1. Background
2. Our results
3. A taste of techniques

1 Background

simple and fast derandomization

Randomness in computation

› context

- We need randomness for crypto, learning, sublinear-time…
- Conjecture: We don’t need randomness to efficiently
    1. solve decision problems
    2. solve “verifiable” search problems
- Question stands at the heart of complexity theory

BPP $\stackrel{?}{=}$ P

› historic recap

- BPP formally defined in [Gill'77]
- Immediately conjectured to "sort-of" equal P

> We believe that for the unrelativized classes of Turing machines, only speedups for infinitely many inputs can be achieved by probabilistic machines.

BPP $\stackrel{?}{=}$ P

› historic recap

› … in fact, paper even raises stronger conjecture:

> **Conjecture:** If f is a recursive function computed in time T* by some probabilistic Turing machine with error probability bounded away from 1/2 , then there is a deterministic Turing machine which computes f in time O(T*(x)) for infinitely many x.

1 the original paper probably means “infinitely-many input lengths”

Hardness-to-randomness

- more recent history

  - Hard functions ⇒ efficient pseudorandomness
    [Yao,'82, BM'84]
  - Hard functions ⇒ derandomization of BPP
    [NW'94, IW'99, STV'01, SU'01, Uma'03, and others]
  - Conditioned on lower bounds, we have an answer

Hardness-to-randomness

- more recent history

  - Smooth trade-off from [Uma'03][1]:

    $\mathrm{DTIME}[2^n] \not\subseteq \mathrm{ioSIZE}[s] \quad \Rightarrow \quad \mathrm{BPP} \subseteq \mathrm{DTIME}[\approx 2^{s\wedge\text{-}1(n)}]$

  - Extreme point [IW'99]:

    $\mathrm{DTIME}[2^n] \not\subseteq \mathrm{ioSIZE}[2^{.01n}] \quad \Rightarrow \quad \mathrm{BPP} = \mathrm{P}$

    $\Rightarrow \quad \mathrm{BPTIME}[T] \subseteq \mathrm{DTIME}[T^{O(1)}]$

1 we'll revisit the smoothness of this trade-off later (as well as the "≈" sign...)

Superfast derandomization

- … snap back to now

- Doron, Moshkovitz, Oh, and Zuckerman (FOCS 2020) recently asked: Can we do it faster?

  $\mathrm{BPTIME}[T] \subseteq \mathrm{DTIME}[T^c]$ for a small c?

- Classical results can yield “reasonable” c when scaled-up
- Diff between (say) c = 10 and c = 3 is substantial !

Superfast derandomization

- … snap back to now

- What is the actual cost of simulating randomness?
  - new area to explore
  - theoretical basis not formed yet
- The obvious “end-goal” question:

Can we simulate randomness with no cost?

Superfast derandomization

- … snap back to now
  - <u>Main result of [DMOZ'20]:</u>

$$\mathbf{BPTIME}[T] \subseteq \mathbf{DTIME}[\,T^{2.01}\,]$$

conditioned on

$$\mathbf{DTIME}[2^n] \not\subseteq {}_{\mathbf{io}}\mathbf{MASIZE}[\,2^{.99n}\,]$$

Superfast derandomization

› … snap back to now

› Takeaways:

1. Super-fast derand possible, under assumptions!
2. Hypothesis is “too strong”

Superfast derandomization

› ... snap back to now

› Takeaways:

1. Super-fast derand possible, under assumptions!
2. Hypothesis is “too strong”

	hypothesis: func in DTIME[2^n] hard for	conclusion: BPTIME[T] in DTIME
[IW’99]	SIZE[$2^{.01n}$]	$T^{O(1)}$
[DMOZ’20]	MASIZE[$2^{.99n}$]	$T^{2.01}$

2 Our results

simple and fast derandomization

Derandomization with no overhead

› our first main result

› Thm 1:

$$\mathbf{BPTIME[T]} \subseteq \mathbf{DTIME[}\, n \cdot T^{1.01} \,\mathbf{]}$$

conditioned on

one-way functions &
non-uniformly-strong time hierarchy

Derandomization with no overhead

› our first main result

› Thm 1:

$$\mathbf{BPTIME[T]} \subseteq \mathbf{DTIME[\, n \cdot T^{1.01}\,]}$$

conditioned on

one-way functions (non-uniformly secure) **&**

$\mathbf{DTIME[2^{kn}]} \not\subset \mathbf{{}_{io}DTIME[2^{.99k \cdot n}]/2^{.99n}}$ (k = suff. large const)

1 the precise technical form of the second hypothesis (a relaxed form) is necessary for obtaining the conclusion using the standard approach of PRGs

Derandomization with no overhead

› meaning of the main result

› Takeaways:

1. Derandomization with near-linear overhead possible!
   (under reasonable assumptions)

2. Hypothesis more standard than in [DMOZ'20]

3. Theoretical basis for superfast derandomization

Derandomization with no overhead

› meaning of the main result

- Randomness might be nearly useless for large T
  - our derandomization is simple & also solves search problems
  - time overhead is minor (to be improved in 2 slides)
- Derandomize "better-than-brute-force" algorithms
- Lower bds for DTIME ⇒ lower bds for BPTIME
  - SETH ⇒ rSETH (assuming Thm 1 for arbitrarily small savings)

Simple and intuitive proofs

- understanding superfast derandomizatoin

- Proof of Thm 1 is intuitive and technically non-involved
  - combining new insights with known technical tools
- In addition: Simple proof for the [DMOZ’20] result
  - extends it: relaxed hardness ⇒ cubic/quartic derand

Complementing the first result

› zooming-in on the precise overhead

› Thm 1': Under assumptions mildly stronger / more involved:

$$\mathbf{BPTIME[T]} \subseteq \mathbf{DTIME[\, n^{1.01} \cdot T \,]} \quad (\forall\ T \leq \text{subexp})$$

Complementing the first result

› zooming-in on the precise overhead

› Thm 1': Under assumptions mildly stronger / more involved:

$$\mathbf{BPTIME[T]} \subseteq \mathbf{DTIME[\,n^{1.01} \cdot T\,]} \quad (\forall\ T \leq \text{subexp})$$

› Thm 2: Conditioned on #NSETH,

$$\mathbf{BPTIME[T]} \not\subseteq \mathbf{DTIME[\,n^{.99} \cdot T\,]} \quad (\forall\ T = \text{poly})$$

› #NSETH: We cannot count solutions for a given k-SAT formula in NTIME[$2^{(1-\varepsilon)\cdot n}$] (assuming suff. large $k=k_\varepsilon$)

Average-case derandomization

› bypassing this barrier

› Thm 3: Under assumptions very similar to Thm 1:

BPTIME[T] ⊆ DTIME[$n^{0.01}$ · T] in average-case

› ... with respect to all T-time samplable distributions

› ... with success probability $1\text{-}n^{-\omega(1)}$

› L ∈ BPTIME[T] ⇒ one alg A_L "looks correct" to all T-time dist.

Extra goodies in the paper

- technical insights intertwined in our proofs

1. Easy way to bypass an informidable-looking barrier
2. Batch-computable PRGs vs amortized time-complexity
3. General simplification of a well-known PRG paradigm
   - "extract-from-pseudoentropic string" as a special case of an easy-to-analyze strategy
4. New light on "hybrid argument" barrier (it's not the one above)

3 A taste of techniques

observations & proof sketches

Technical roadmap

› what we'll talk about

› Bypassing the seed-length barrier

› Proof sketch for Thm 1

› Simplifying a well-known PRG paradigm

Bypassing the seed-length barrier

one technical observation to remember

Hardness-to-randomness

› the basic idea

› hard functions ⇒ random-looking bits [Yao,'82, BM'84]

$$\underbrace{\boxed{x_1}\quad \boxed{x_2}\quad \boxed{x_3}\quad \cdots\quad \boxed{x_n}}\quad \boxed{f(x)}$$

› f hard on average ⇒ ($\mathbf{x}\circ f(\mathbf{x})$) looks random

Hardness-to-randomness

› reconstructive PRGs

PRG

G^f: $\ell(n)$ $\Rightarrow$ $T(n)$

Hardness-to-randomness

› reconstructive PRGs

distinguisher

PRG

G^f: $\boxed{\ell(n)} \Rightarrow \boxed{T(n)}$

Hardness-to-randomness

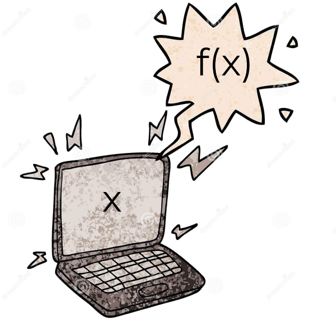
› reconstructive PRGs

PRG based on
hard function f

G^f: $\ell(n)$ $\Rightarrow$ $T(n)$

Hardness-to-randomness

› reconstructive PRGs



distinguisher
yields efficient
procedure for f

G^f: $\boxed{\ell(n)} \Rightarrow \boxed{T(n)}$

Hardness-to-randomness

› derandomization from PRGs

› replace T(n) coins with $\ell(n)$ coins, enumerate in time $2^{\ell(n)}$

› textbook results [NW'94,IW'99,STV'01,SU'01,Uma'03]:

$$\boxed{\textbf{DTIME}[2^n] \not\subset \textbf{SIZE}[s]} \Rightarrow \boxed{\textbf{PRG with stretch} \approx s} \Rightarrow \boxed{\textbf{BPP in DTIME} \approx 2^{s\text{^-1}(n)}}$$

An informidable-looking barrier

- why experts might think that c<2 requires "new techniques"

- Textbook approach: To derandomize time-T algs, construct a PRG that fools non-uniform size-T circuits
- Such a PRG requires a seed of length log(T)
- The derandomization time is $2^{\log(T)} \cdot T(n) \geq T(n)^2$

Tracking the non-uniformity

› modeling distinguishers, carefully

who is this distinguisher?

T(n)

Tracking the non-uniformity

› modeling distinguishers, carefully

› For any L ∈ BPTIME[T], our focus is:

Does the probabilistic machine M_L behave the same on $\mathbf{G^f(u_{\ell(n)})}$ & $\mathbf{u_{T(n)}}$ for all inputs x?

› Distinguisher is M_L with an arbitrary fixed input x

T(n)

Tracking the non-uniformity

› modeling distinguishers, carefully

› Textbook distinguisher:

Non-uniform circuit of size T

› Our pivotal observation:

Distinguisher is a time-T machine with $|n| \ll T$ bits of non-uniformity

T(n)

Why is this helpful?

› fooling small non-uniformity with small seed length

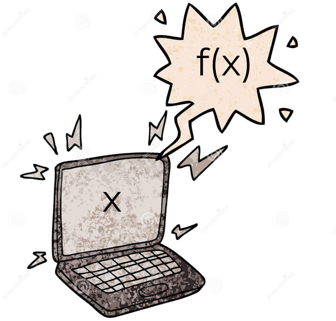
- non-uniformity is $n \ll T(n)$
- exists non-explicit PRG with seed length $\log(n) \ll \log(T)$!
- opens the door to derandomization in time $n \cdot T(n)$
  - we just to make this PRG explicit, under assumptions

Proof sketch for Thm 1

main ideas & some parameters

Hardness-to-randomness

› reconstructive PRGs



distinguisher
yields efficient
procedure for f

G^f: $\boxed{\ell(n)}$ ⇒ $\boxed{T(n)}$

Reconstruction overhead

› and its discontents

› “Reconstructive” procedure for f as oracle machine

$$\text{Reconst}^{\text{DISTINGUISHER}}(x) = f(x)$$

› Reconstruction overhead is the main bottleneck

› Inefficient reconstruction

$\Rightarrow$ inefficient procedure for f

$\Rightarrow$ stronger hardness hypothesis

Reconstruction overhead

- and its discontents

- Best known overhead [Uma'03]:

  distinguisher in time $T \Rightarrow$ procedure for f in time $T^{O(1)}$

- … so we need to assume f is hard for time $T^{O(1)}$
- … since the PRG computes $f \Rightarrow$ PRG takes time $\geq T^{O(1)}$
- Derandomization with large polynomial overhead

Our PRG construction

› high-level overview

- Our goal is to avoid this overhead
- Two ideas in the proof:
  1. Compose two PRGs computable in time $\approx T^{1.01}$
  2. Use a tiny & super-exponentially-hard truth-table

Composing two “low-cost” PRGs

- each computable in time $\approx T^{1.01}$

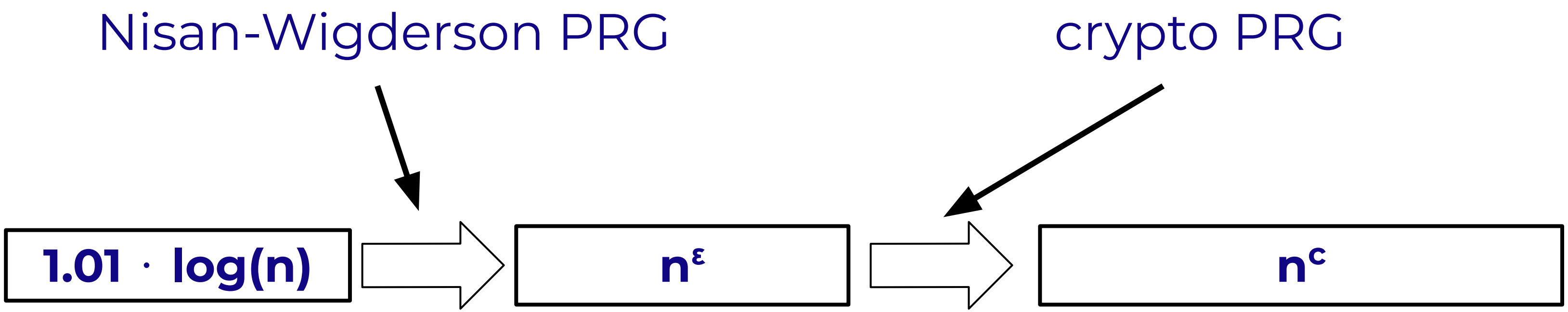
  - Focus on $T(n) = n^c$ for simplicity

$$\boxed{\mathbf{1.01 \cdot \log(n)}} \Rightarrow \boxed{\mathbf{n^{\varepsilon}}} \Rightarrow \boxed{\mathbf{n^{c}}}$$

Composing two “low-cost” PRGs

- each computable in time ≈ $T^{1.01}$

  - Focus on $T(n) = n^c$ for simplicity



Composing two “low-cost” PRGs

› the “inner” PRG

› Small seed, but small output length

› <u>Obs:</u> Small output length ⇒ small reconst. overhead

Nisan-Wigderson PRG

$$\boxed{\mathbf{1.01 \cdot \log(n)}} \Rightarrow \boxed{\mathbf{n^{\varepsilon}}}$$

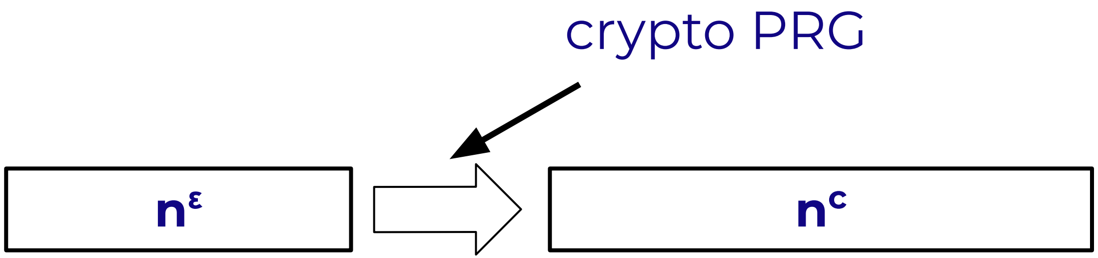
Composing two “low-cost” PRGs

› the “outer” PRG

› Large output length, but large seed

› <u>Obs:</u> OWF ⇒ crypto PRG ⇒ near-linear time PRG[1]

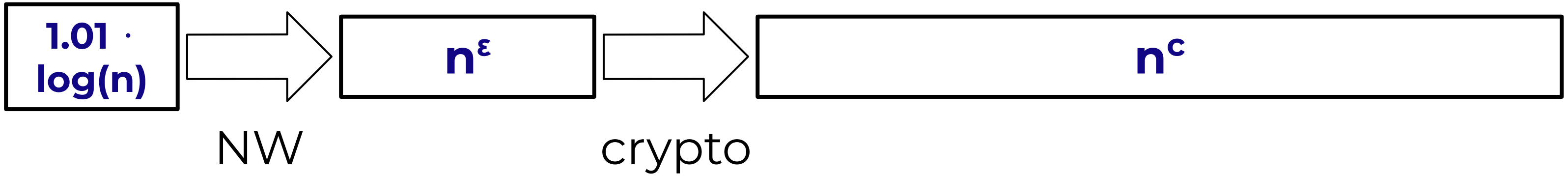
  › we’ll use it as a non-crypto PRG, i.e. distinguisher is weaker



1 take PRG $n^{\varepsilon} \mapsto 2n^{\varepsilon}$ computable in time $n^{O(\varepsilon)}$, and “compose” it $\approx n^{c}$ times to extend output to n^{c}

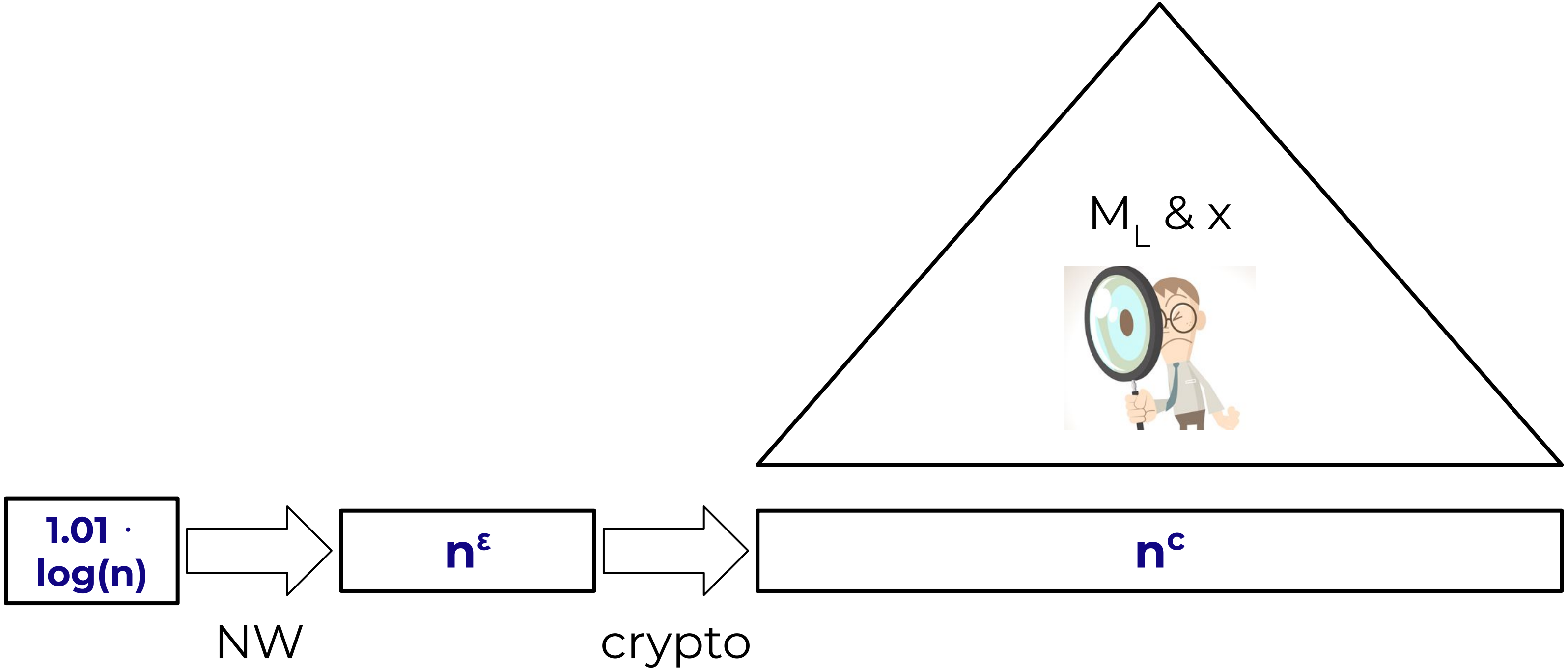
Tiny, superexp-hard truth-table

› a parametric overview



Tiny, superexp-hard truth-table

› a parametric overview



Tiny, superexp-hard truth-table

› a parametric overview

› Our distinguisher uses

1. time n^c
2. advice n

M_L & x

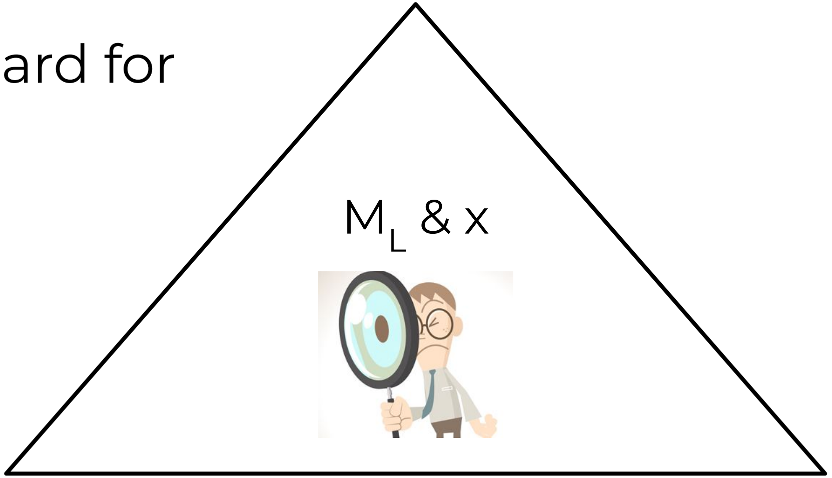
$1.01 \cdot \log(n)$ $\xrightarrow{\text{NW}}$ n^{ε} $\xrightarrow{\text{crypto}}$ n^c

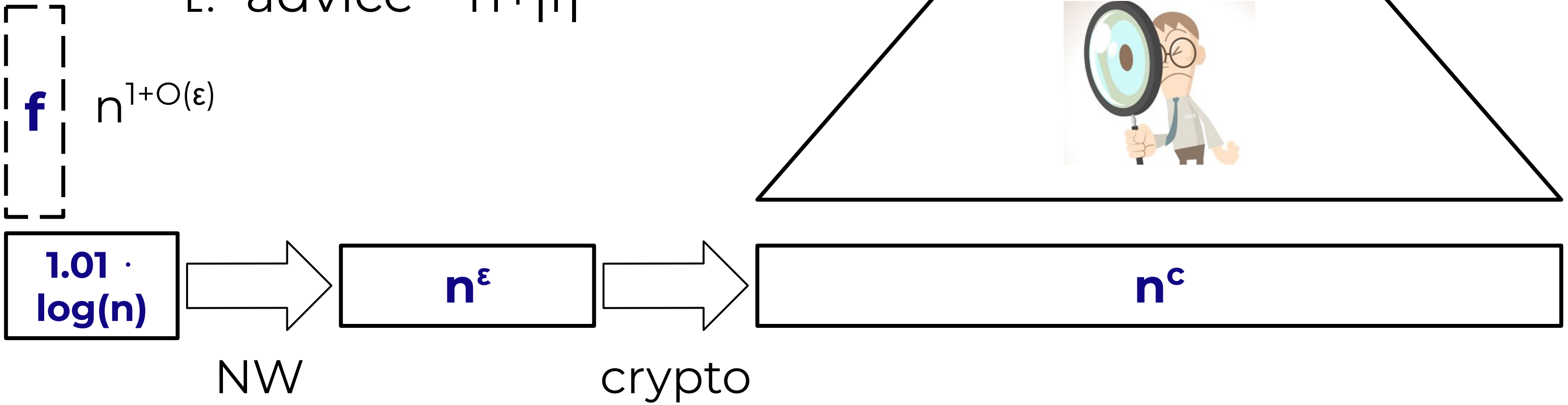
Tiny, superexp-hard truth-table

› a parametric overview

› We use $|f| = n^{1+O(\varepsilon)}$ that is hard for

1. time $n^{1.01 \cdot c}$
2. advice $n+|f|^{0.99}$





Tiny, superexp-hard truth-table

› a parametric overview

› We use $|f| = n^{1+O(\varepsilon)}$ that is hard for

1. time $n^{1.01 \cdot c} \approx \mathbf{2^{c \cdot \ell}}$
2. advice $n+|f|^{0.99} \approx \mathbf{2^{0.99 \cdot \ell}}$

M_L & x

f $n^{1+O(\varepsilon)} = \mathbf{2^{\ell}}$

1.01 · log(n) $\xrightarrow{\text{NW}}$ $\mathbf{n^{\varepsilon}}$ $\xrightarrow{\text{crypto}}$ $\mathbf{n^{c}}$

Hardness hypothesis

› generalizing classical hardness hypotheses

› Our derandomization uses a tiny truth-table with super-exponential time complexity

› Our hardness hypothesis (for $k \approx c$)

$$f \in \mathrm{DTIME}[2^{k \cdot n}] \text{ and hard for } \mathrm{DTIME}[2^{.99k \cdot n}]/2^{.99n}$$

Hardness hypothesis

› generalizing classical hardness hypotheses

› Natural "scale-up" of classical hypotheses:

$f \in \mathrm{DTIME}[2^{k \cdot n}] \setminus \mathrm{DTIME}[2^{.99k \cdot n}]/2^{.99n}$ [this work]

$f \in \mathrm{DTIME}[2^{n}] \setminus \mathrm{SIZE}[2^{.01n}]$ [IW'99]

› Both interpreted as non-uniformly-strong time-hierarchy

A last small gap

- final running-time of derandomization?

  - we'll have $n^{1.01}$ seeds (for the inner PRG NW)
  - if on each seed PRG is computable in time $\approx T$,
    then we get derandomization in time $O(n^{1.01} \cdot T)$

A last small gap

- we didn't *really* see that the PRG is linear-time computable yet

- PRG isn't computable in time ≈ T on each seed !
- … but it's computable on all seeds in amortized time ≈ T
  - suffices for derandomization
- … and we can relax the hypothesis, and only require f to be computable on all inputs in amortized time ≈ T

A last small gap

- we didn't *really* see that the PRG is linear-time computable yet

- Assuming OWFs, tight equivalence of
  1. batch-computable PRGs
  2. hard functions with small amortized time-complexity
- The "right" objects to study in hardness-to-randomness
  - the tightness is significant for superfast derandomization

Zooming-in on the overhead

› a reminder of additional results in the paper

› <u>Thm 1':</u> Reduce overhead to $n^{1.01} \cdot T$, extend to $T(n) = n^{\omega(1)}$

› <u>Thm 2:</u> Assuming #NSETH, overhead of $n^{.99} \cdot T$ is optimal

› <u>Thm 3:</u> Average-case derandomization with effectively no overhead at all (only n^{ε}, below lower bound)

Simplifying a well-known PRG paradigm

via quantified derandomization

Well-known PRG paradigm

- underlies [HILL’99, BSW’03, …, DMOZ’20]

- Paradigm for constructing PRGs
- Based on composition of two algorithms
- We will show: Any such composition can be viewed & analyzed in a very simple way

Well-known PRG paradigm

› underlies [HILL'99, BSW'03, ..., DMOZ'20]

1. a pseudoentropy generator (PEG)

Well-known PRG paradigm

› underlies [HILL’99, BSW’03, …, DMOZ’20]

1. a pseudoentropy generator (PEG)

G^f: $\boxed{\ell(n)} \Rightarrow \boxed{T(n)}$

Well-known PRG paradigm

› underlies [HILL'99, BSW'03, ..., DMOZ'20]

1. a pseudoentropy generator (PEG)

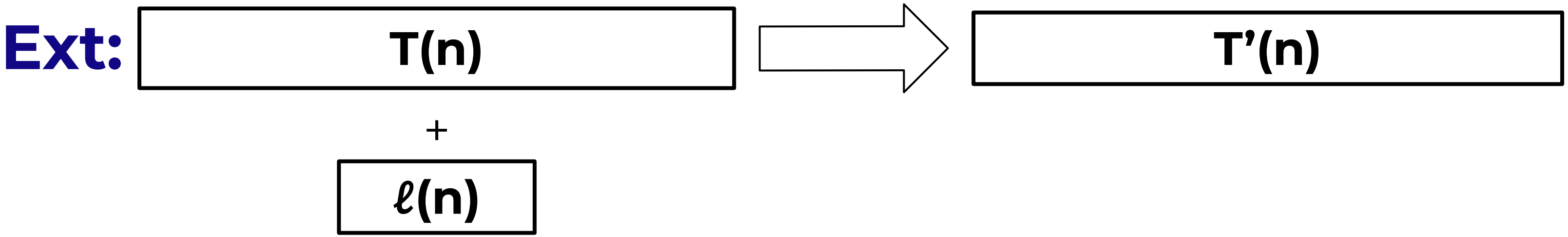
wow, this has high entropy!

G^f: $\boxed{\ell(n)} \Longrightarrow \boxed{T(n)}$

Well-known PRG paradigm

› underlies [HILL'99, BSW'03, …, DMOZ'20]

1. a pseudoentropy generator (PEG)
2. a randomness extractor

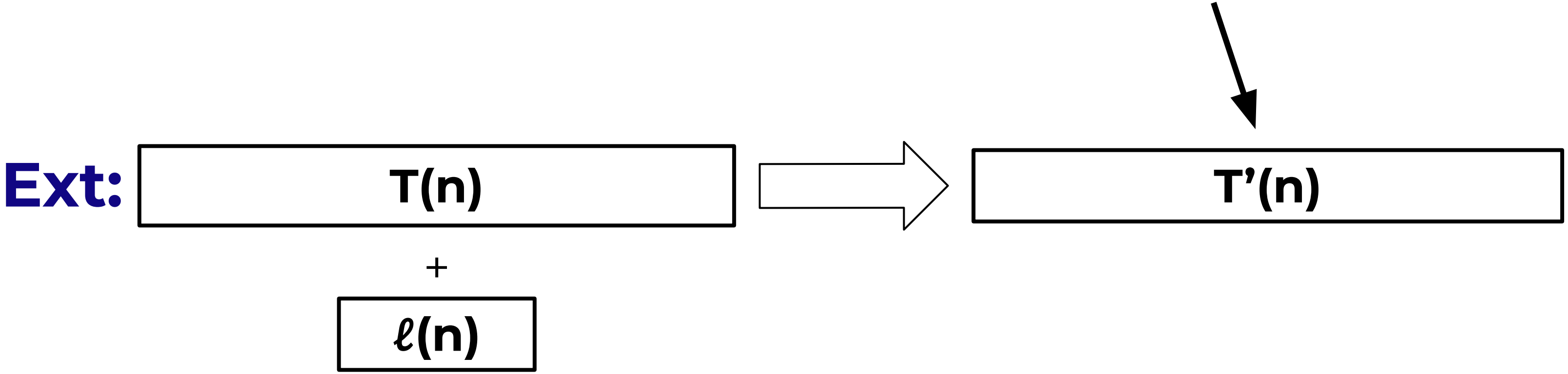


Well-known PRG paradigm

› underlies [HILL'99, BSW'03, …, DMOZ'20]

1. a pseudoentropy generator (PEG)
2. a randomness extractor

all the entropy "extracted"
to almost-uniform string



Well-known PRG paradigm

- underlies [HILL'99, BSW'03, ..., DMOZ'20]

- PRG: **$G(s_1, s_2) = \mathrm{Ext}(\ \mathrm{PEG}(s_1), s_2\ )$**
- Intuition: If $\mathrm{PEG}(s_1)$ looks entropic, then $\mathrm{Ext}(\ \mathrm{PEG}(s_1), s_2\ )$ should look random

- Good extractors are known, so we "just" need a PEG, and to make the idea above for composition work

The main challenge

› and the approach of [DMOZ'20]

- Constructing PEGs challenging, only few known ideas
- The approach of [DMOZ'20]:
  1. Construct "metric" (weak) PEG from hard funcs; but
  2. Highly non-trivial composition with extractor
     - requires PEG to fool strong & non-standard class

Easier & more general paradigm

- error-reduction then quantified derandomization
  - PRG: $\mathbf{G(s_1, s_2) = Ext(\,PEG(s_1), s_2\,)}$

Easier & more general paradigm

- error-reduction then quantified derandomization

- PRG: $\mathbf{G(s_1, s_2) = Ext(\ PEG(s_1), s_2\ )}$
- We show a simpler analysis of the composition
- … which paves way to using a weaker “inner” generator

Easier & more general paradigm

- error-reduction then quantified derandomization

- PRG: $\mathbf{G(s_1, s_2) = Ext(PEG(s_1), s_2)}$
- Our analysis involves two steps:
  1. (non-standard) error reduction, using Ext
  2. quantified derandomization, using the "inner" generator

Easier & more general paradigm

- error-reduction then quantified derandomization

- PRG: $\mathbf{G(s_1, s_2) = Ext(\ QD(s_1), s_2\ )}$
- Generators for QD are equivalent to metric ("weak") PEGs
- … but no need now for generator to fool a stronger class
- And analysis of composition is very simple

Easier & more general paradigm

- error-reduction then quantified derandomization

- <u>Prop:</u> Any construction that can be analyzed as “extract from a pseudoentropic string” can also be analyzed as “non-standard error-reduction and QD”
  - (converse not known)
- To materialize this approach we need a generator for QD
  - not an easy task, in general

Derandomization with overhead $c \in \{2,3,4\}$

- easy & versatile proof for superfast derandomization

- Prop: Assuming $\mathsf{DTIME}[2^n] \not\subset {}_{\mathsf{io}}\mathsf{MASIZE}[2^{.99n}]$, there exists a very simple & fast generator for QD
- Cor: New and easy proof for main result of [DMOZ'20]
  - pf: combine QD construction with easy high-level analysis
- Proof versatile, extends to cubic/quartic derandomization from hardness only for NSIZE (details in paper)

4 Key takeaways

results to remember

Take-home message

1. Derandomization with overhead $\approx n \cdot T(n)$ possible under reasonable assumptions
2. Simple & intuitive proofs yield conditional derandomization with overhead $c \in \{1,2,3,4\} + \varepsilon$
3. Broadening the theoretical basis for superfast derandomization

Take-home open questions!

› a sample of some directions

1. Is the overhead of $n \cdot T$ optimal?
   › evidence without #NSETH
2. Search-to-decision with minimal overhead?
   › true given OWFs, show unconditional reduction
3. Superfast derandomization from classical hypotheses?
   › no crypto, no hardness for MASIZE/NSIZE

Thank you!

$\Rightarrow$ derandomization with essentially no overhead

$\Rightarrow$ simple & intuitive proofs relying on new high-level insights