Hardness Self-Amplification: Simplified, Optimized, and Unified

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joint work with Shuichi Hirahara

Average-Case Complexity

- **How many hard instances?**
  - complexity of **random** instance

- **Motivation**: pessimism of worst-case complexity, derandomization, crypto

Reductions in worst-case hardness:

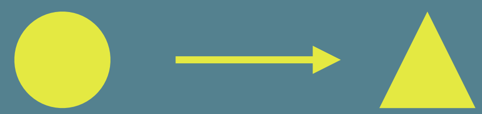
▲ may have “structure” due to the gadget construction.
→ hardness of “structural” instances

Average-Case Complexity

- **How many hard instances?**
  - complexity of **random** instance

- **Motivation**: pessimism of worst-case complexity, derandomization, crypto

Hardness of unstructured instances?



▲ may have "structure" due to the gadget construction.
→ hardness of "structural" instances

Average-Case Complexity

- **Algo A computes f with success probability γ if** $\Pr_x[A(x) = f(x)] \geq \gamma$
  - x is chosen from some distribution (over inputs of fixed size)
- f **is worst-case hard** $\stackrel{\text{def}}{\iff}$ **$\forall$efficient algo A, $\exists x$, $A(x) \neq f(x)$**
- f **is strongly-hard** $\stackrel{\text{def}}{\iff}$ **$\forall$efficient algo has success prob ≤ 0.01**

$$\boxed{f \text{ is strongly-hard}} \quad \underset{\text{strong random self-reduction}}{\overset{\text{trivial}}{\rightleftarrows}} \quad \boxed{f \text{ is worst-case hard}}$$

Average-Case Complexity

- **Algo A computes f with success probability γ if** $\Pr_x[A(x) = f(x)] \geq \gamma$
  - x is chosen fro
- f **is worst-case ha**
- f **is strongly-hard**

possible for some problems!

Permanent **[Cai, Pavan, Sivakumar, 1999]**
Matrix multiplication **[Asadi, Golovnev, Gur, Shinkar, 2022]**
Discrete logarithm (folklore)
etc…

t al

f is strongly-hard ⟶ **f is worst-case hard**

f is strongly-hard ⟵ **f is worst-case hard**

strong random self-reduction

Average-Case Complexity

- **Algo A computes f with success probability γ if** $\Pr_x[A(x) = f(x)] \geq \gamma$
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- f

possible for some problems!

Permanent **[Cai, Pavan, Sivakumar, 1999]**

Matrix multiplication **[Asadi, Golovnev, Gur, Shinkar, 2022]**

What problems admit strong RSR?

t al

f is strongly-hard ⇄ **f is worst-case hard**

strong random self-reduction

Hardness Self-Amplification

- f is **weakly-hard** $\overset{\text{def}}{\iff}$ $\forall$efficient algo has success prob ≤ 0.99

our topic

$$\begin{tikzcd} \text{f is strongly-hard} & \text{f is weakly-hard} \arrow[l, "\text{hardness self-amplification}"'] & \text{f is worst-case hard} \arrow[l, "\text{weak RSR}"'] \end{tikzcd}$$

Our Results

- **Our Paper: hardness self-amplification for popular problems**
  - matrix multiplication (MM)
  - online matrix-vector problem (OMv)
  - triangle counting (TC)
  - planted clique (PC)

- **Corollary**
  - new strong RSR for MM, OMv, TC
  - search-to-decision reduction of PC
  - improves and simplifies previous RSR for those problems

Our Results

- **Our Paper: hardness self-amplification for popular problems**
  - matrix multiplication (MM)
  - online matrix-vector problem (OMv)
  - triangle counting (TC)
  - planted clique (PC)

- **Our Ingredient**
  - A framework of hardness amplification using **expanders (samplers)**
  - The same framework was previously used to obtain Direct Product Theorem

[Impagliazzo, Jaiswal, Kabanets, Wigderson (2010)]

Matrix Multiplication

Matrix Multiplication (MM)

- **Task: Multiply given $A, B \in \mathbb{F}^{n \times n}$ (finite field $\mathbb{F}$)**
- **Input distribution: Uniform**

<u>Theorem (Blum, Luby, Rubinfeld, 1993)</u>

If we can solve MM with success prob 0.99 in time $T(n)$,
then we can solve MM in time $O(T(n))$ for any input.

weak hardness $\xleftarrow[\text{weak RSR}]{}$ **worst-case hardness**

Matrix Multiplication (MM)

Theorem (Asadi, Golovnev, Gur, Shinkar, 2022)

If we can solve MM with success prob ϵ in time $T(n)$,
then we can solve MM in time $2^{O(\log^5(1/\epsilon))} \cdot T(n)$ for any input.

- strong RSR 😀
- Tool: Additive Combinatorics (quasi-polynomial Bogolyubov-Ruzsa lemma)

strong hardness **weak hardness** [BLR93] **worst-case hardness**

[AGGS22]

Matrix Multiplication (MM)

Theorem (this work)

If we can solve MM with success prob ϵ in time $T(n)$,

then we can solve MM in time $\dfrac{\mathrm{polylog}(1/\epsilon)}{\epsilon} \cdot T(n)$ for any input.

- Proof: **Hardness self-amplification + BLR93**
- Matrices can be over finite **ring**
- **Improved** overhead ($2^{O(\log^5(1/\epsilon))} \to \tilde{O}(1/\epsilon)$)

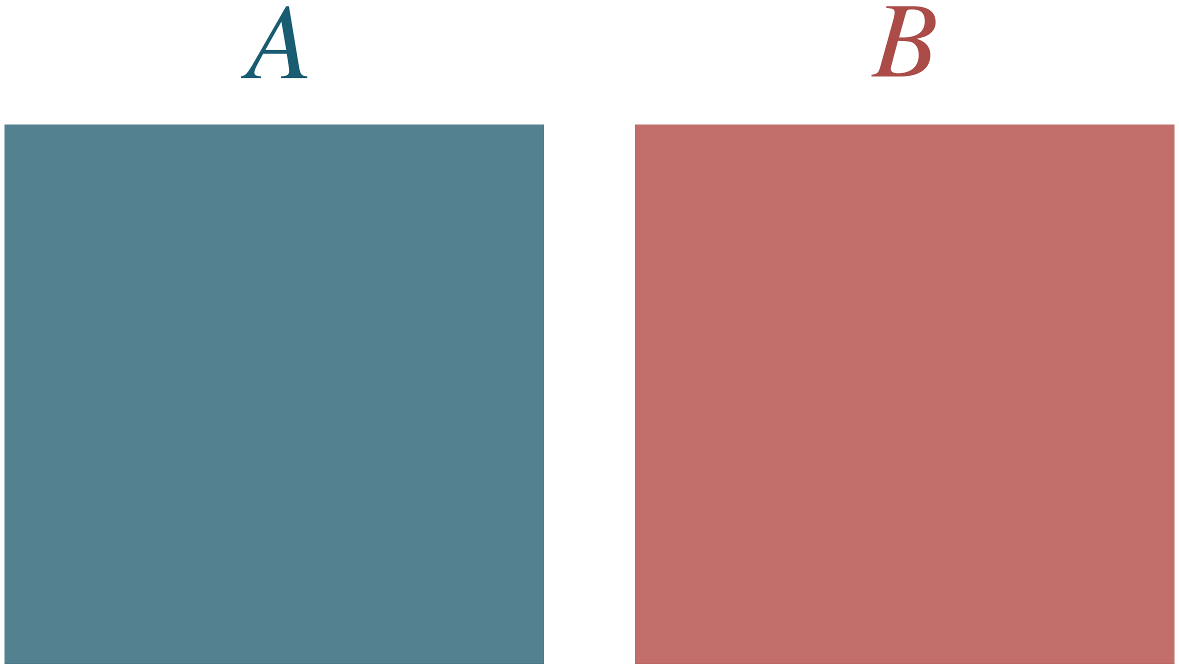
$$\boxed{\textbf{strong hardness}} \xleftarrow{\text{this work}} \boxed{\textbf{weak hardness}} \xleftarrow{\text{[BLR93]}} \boxed{\textbf{worst-case hardness}}$$

Proof Sketch

Assumption: $\exists$ algo $\mathscr{M}$ that solves MM with success prob ϵ

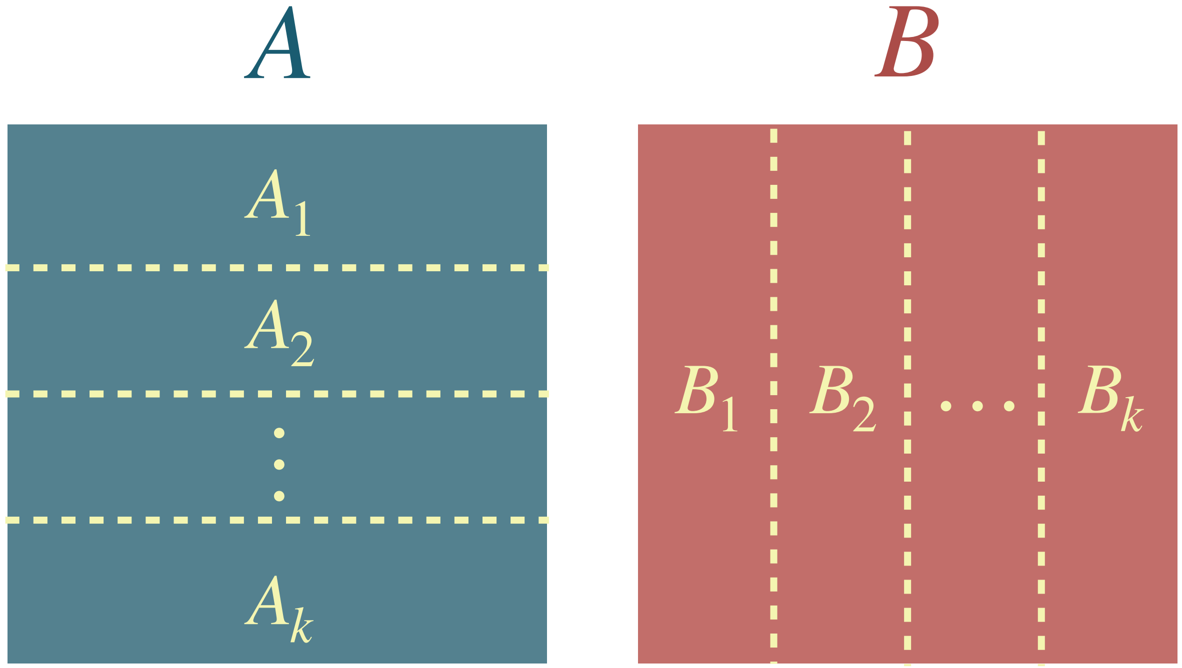
Goal: compute AB for any $A, B \in \mathbb{F}^{n\times n}$

Proof Sketch



Given $A, B \in \mathbb{F}^{n\times n}$ (worst-case instance)

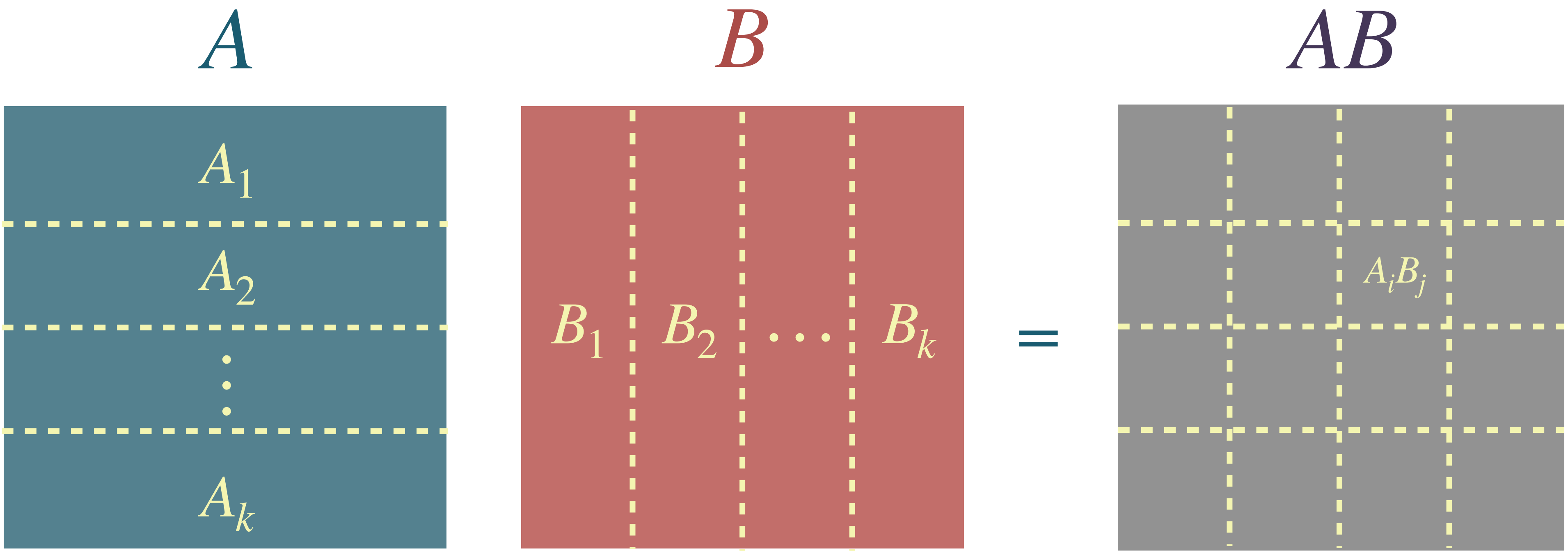
Proof Sketch



Divide A, B into k submatrices

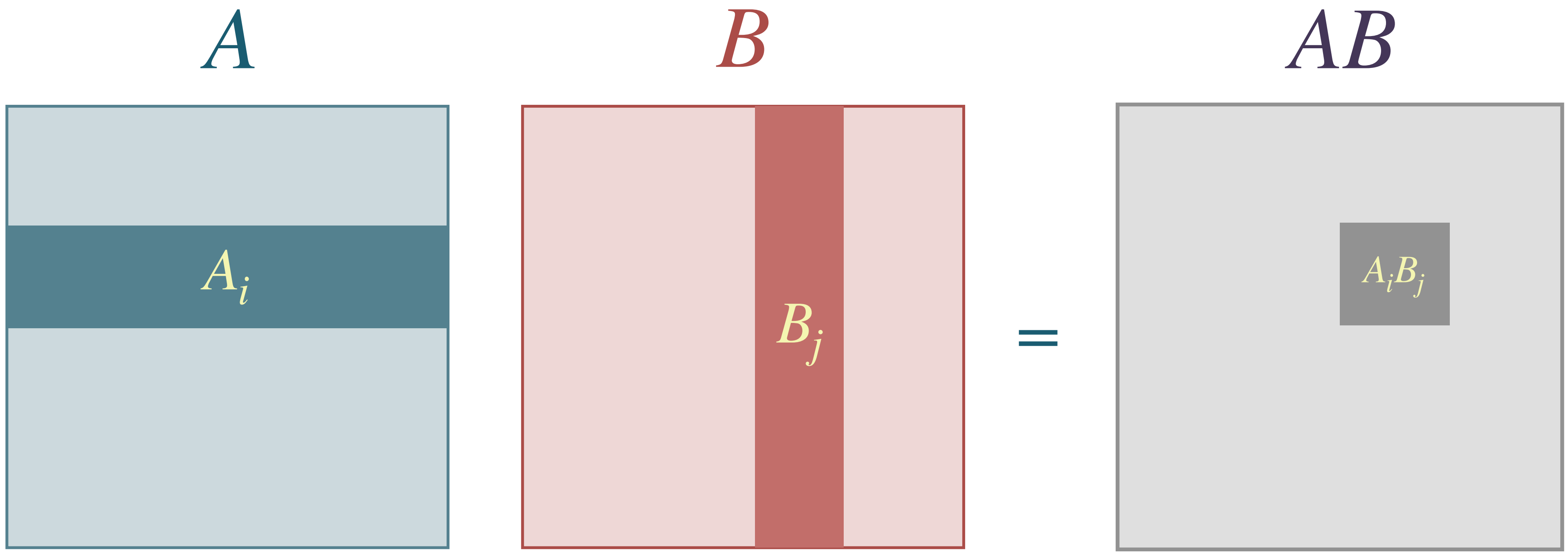
$A_i \in \mathbb{F}^{(n/k)\times n}$ and $B_j \in \mathbb{F}^{n\times(n/k)}$

Proof Sketch



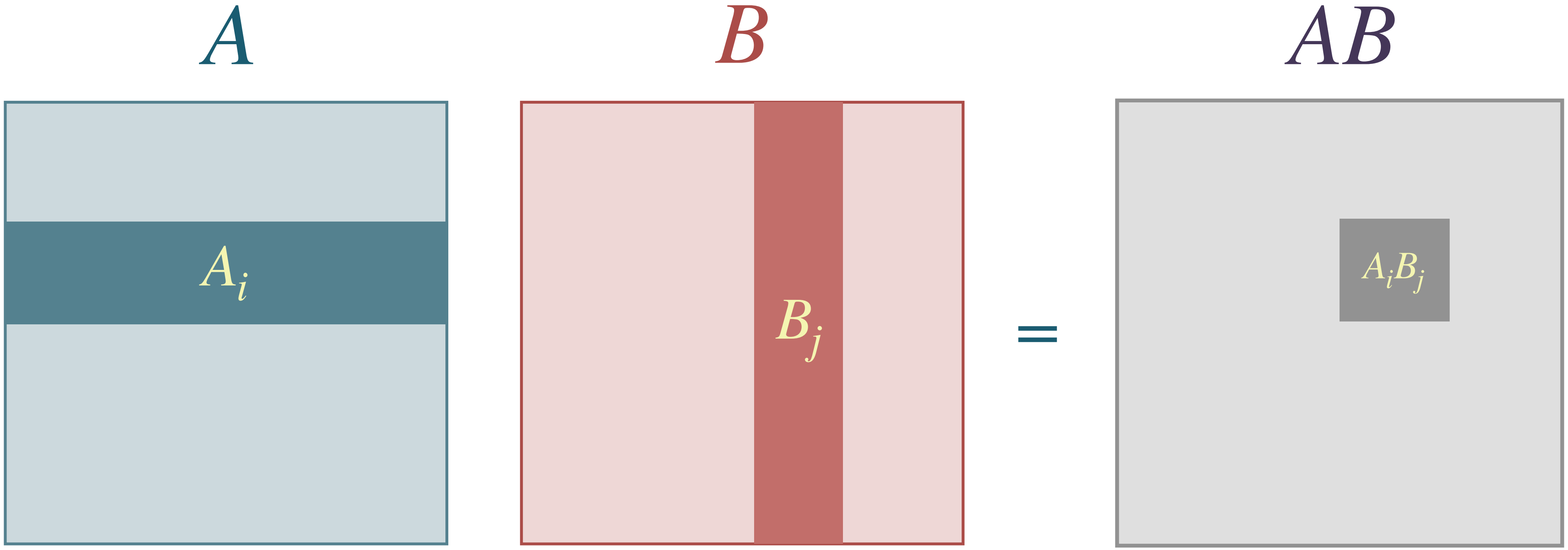
Product AB has $k \times k$ blocks

Proof Sketch



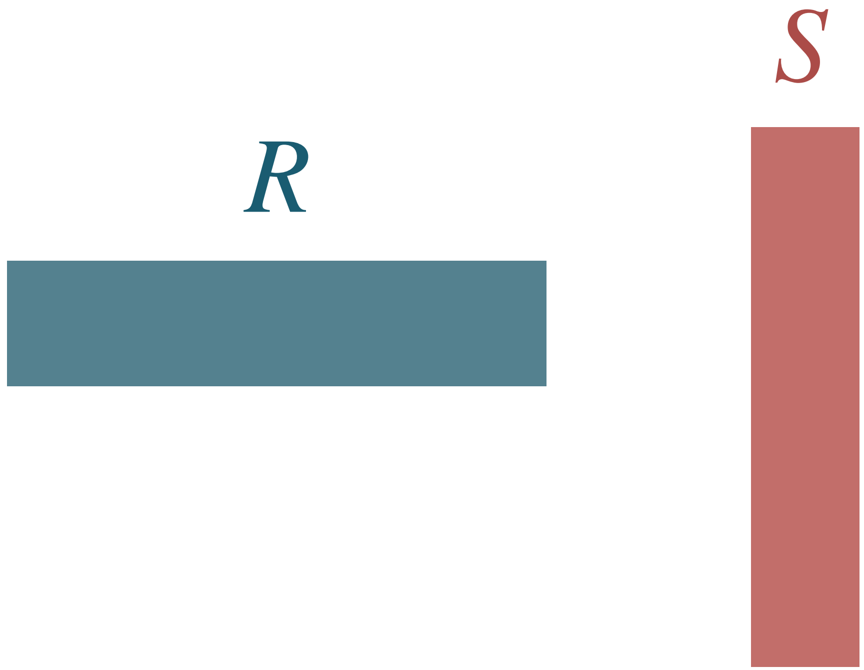
We focus on MM for A_iB_j
(downward self-reduction)

Proof Sketch



By [BLR93], A_i, B_j can be random matrices
(random self-reduction)

Proof Sketch



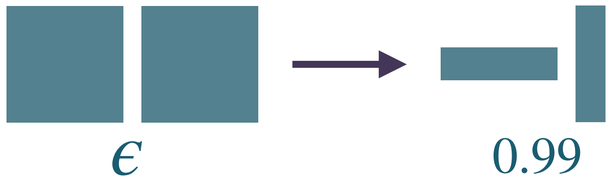
Input: random matrices R, S

Goal: compute RS (with success prob 0.99)

Hardness Amplification

Lemma.

If we can solve MM with success prob ϵ in time $T(n)$,
then we can compute RS with success prob 0.99 for some $k = O(\log(1/\epsilon))$.

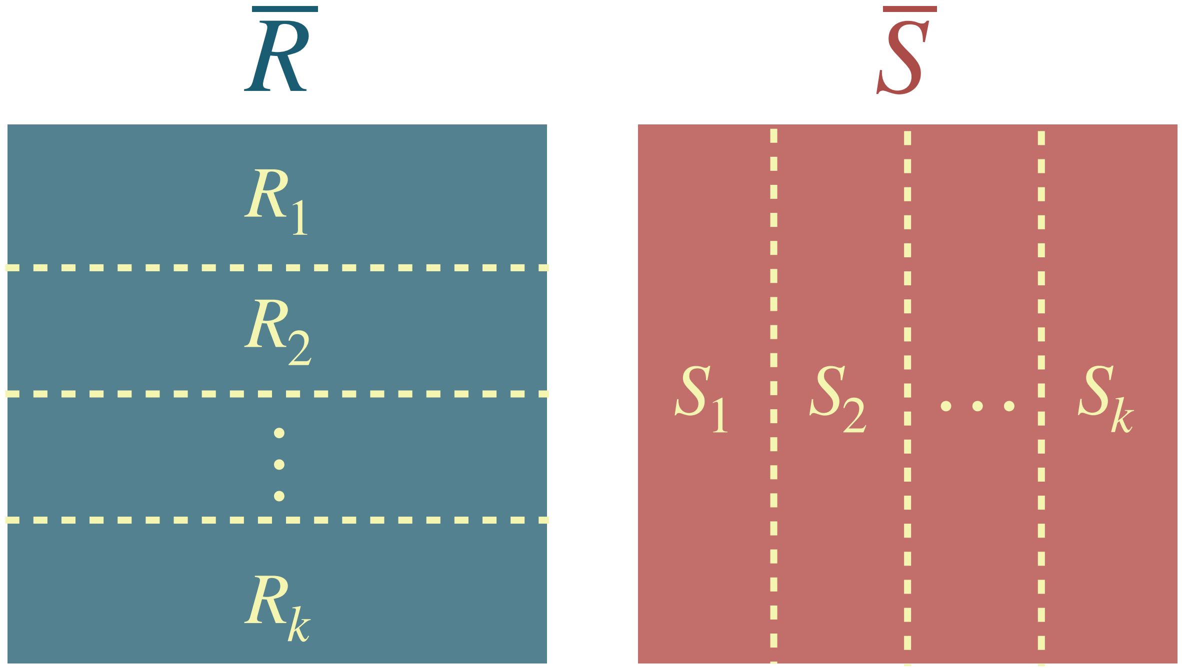


- Hardness (Self-) Amplification for MM
- idea: "**up**ward-reduction"
- Essentially same as the proof of Direct Product Theorem

Proof Sketch

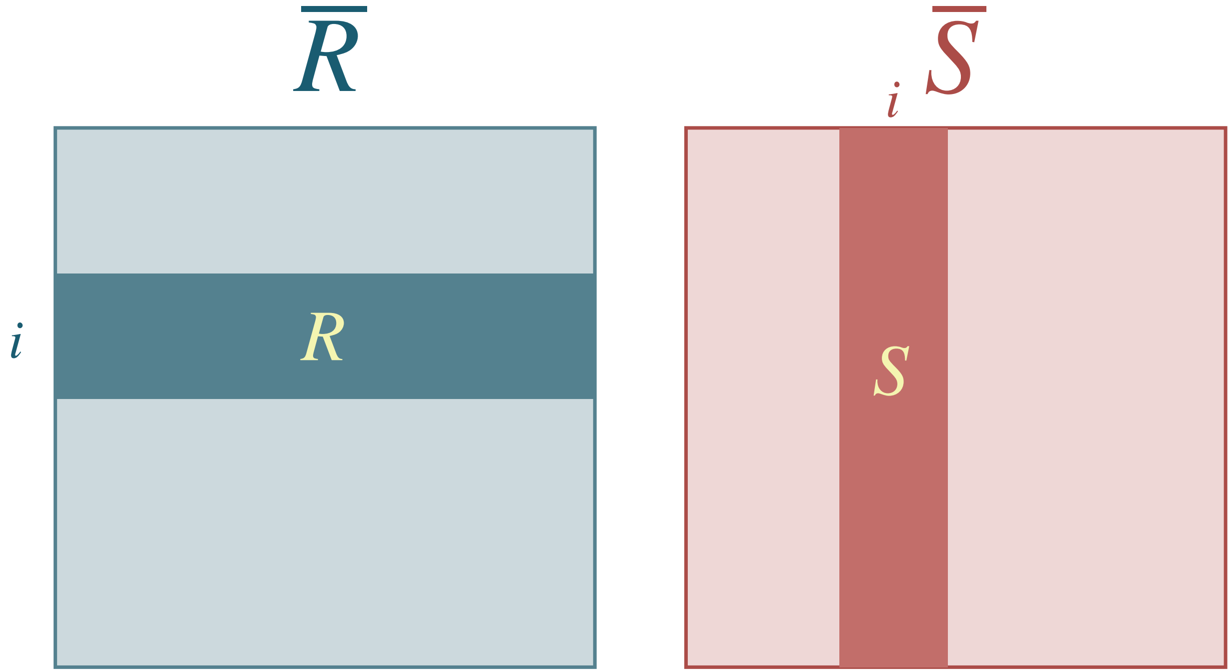
Sample $n \times n$ random matrices $\overline{R}, \overline{S}$

Proof Sketch



Divide R, S into k submatrices : R_i , S_j $(i, j \in [k])$

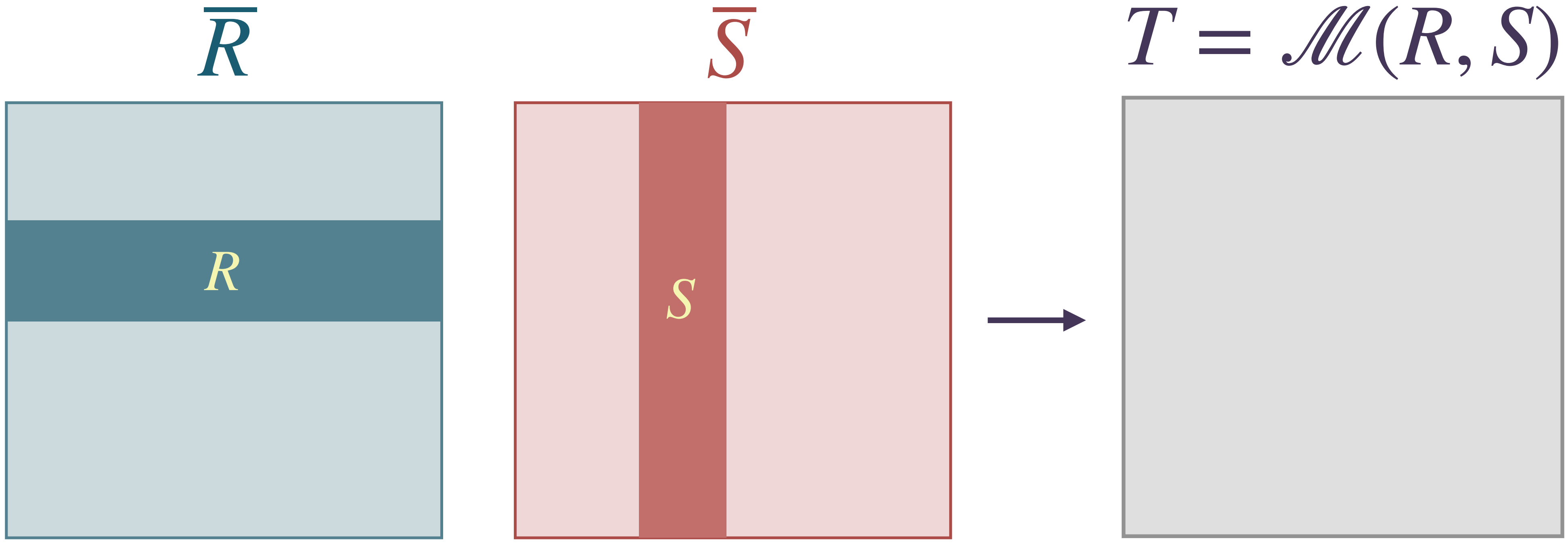
Proof Sketch



Choose random $i \sim [k]$

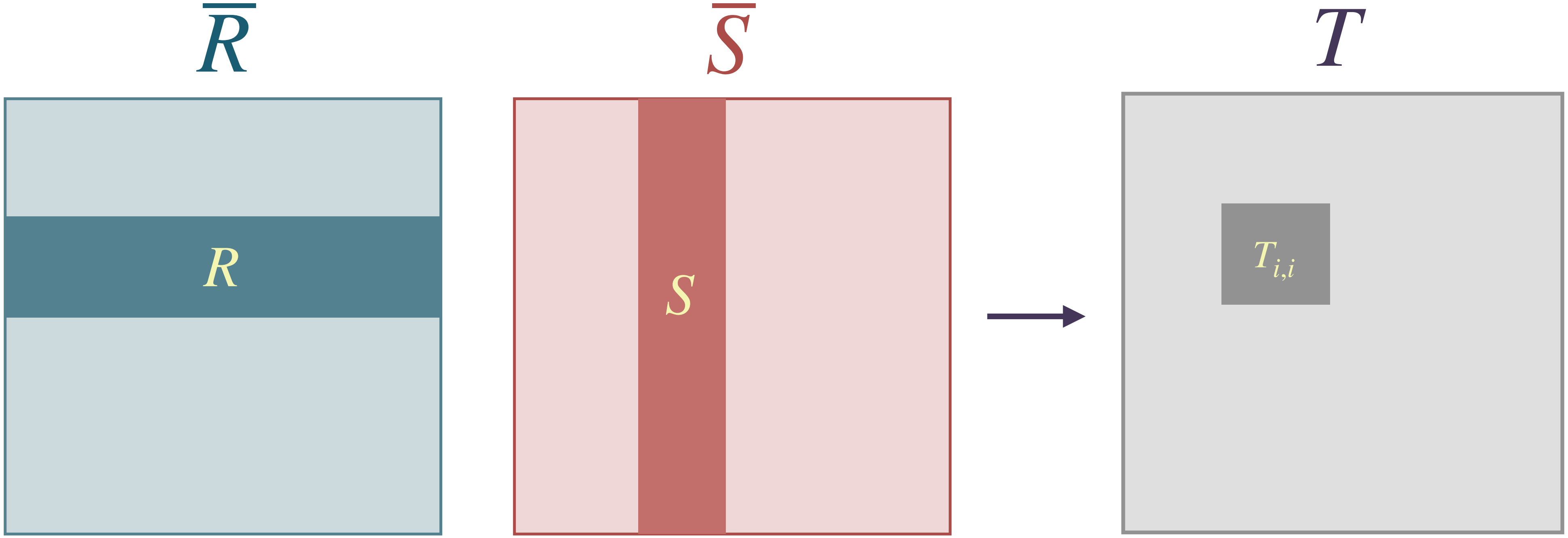
$R_i \leftarrow R$ and $S_i \leftarrow S$

Proof Sketch



Let $\mathscr{R}^{\mathscr{M}}(R, S)$ be the algorithm that outputs $\mathscr{M}(\overline{R}, \overline{S})$

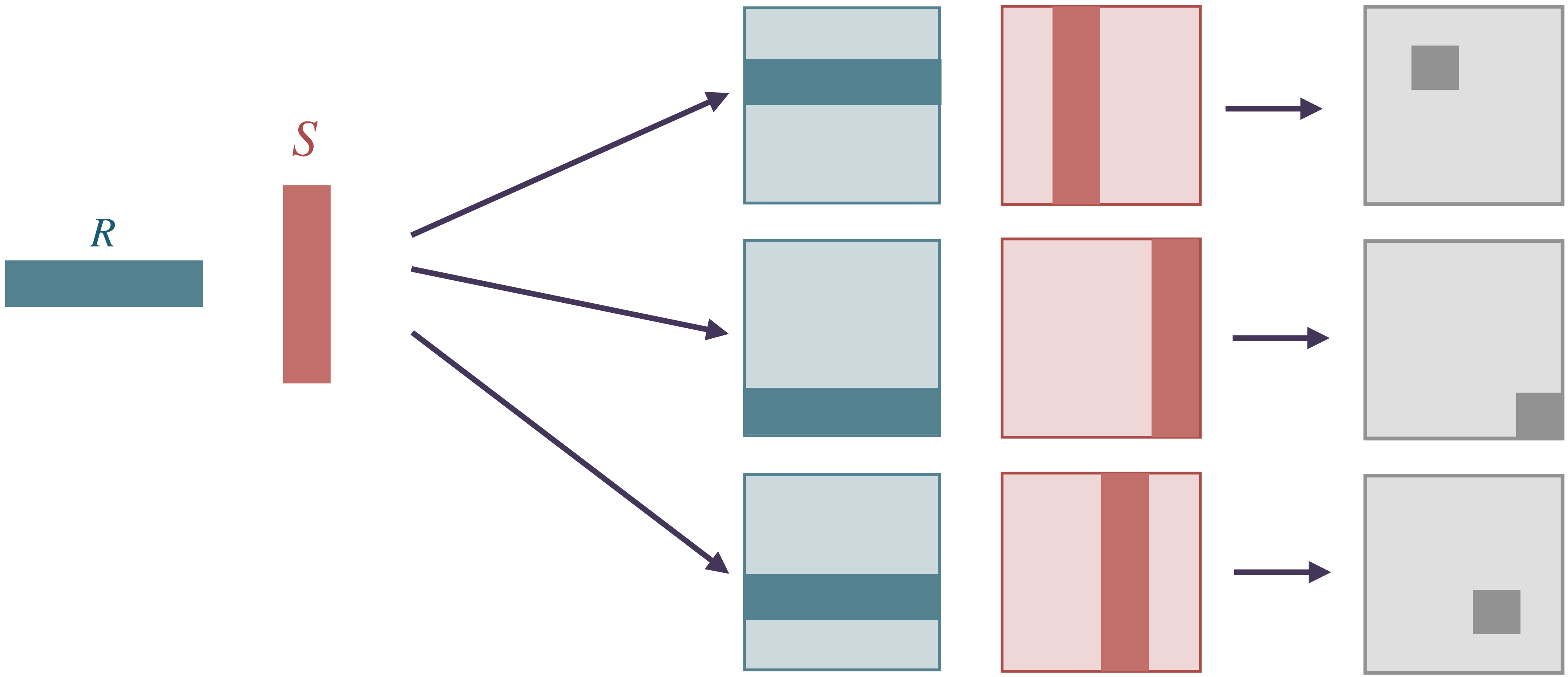
Proof Sketch



Output $T_{i,i}$ if $RS = T_{i,i}$

We can verify in time $O(n^2)$

Proof Sketch



Our algo: Run $\mathscr{R}^{\mathscr{M}}(R, S)$ until we find RS

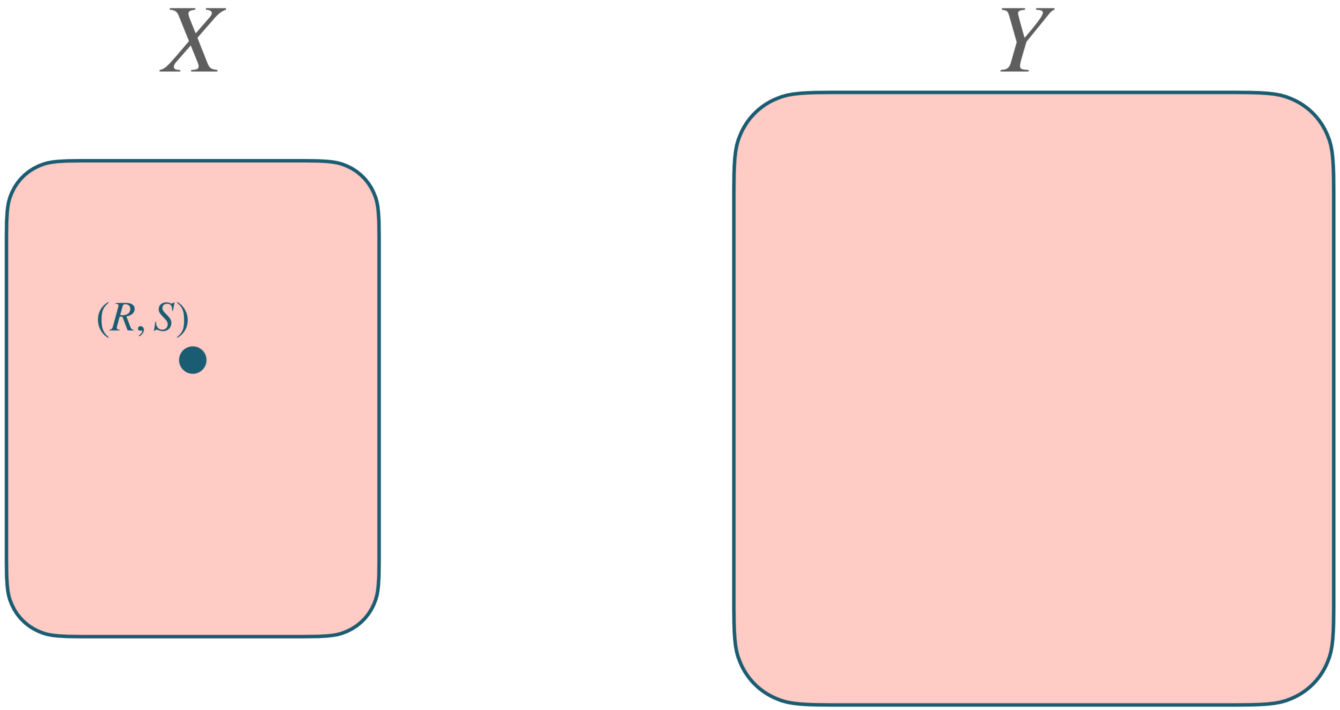
Proof Sketch

Lemma

For 0.99-fraction of (R, S), # of iteration is at most $O(1/\epsilon)$ if $k \geq 100 \log(1/\epsilon)$

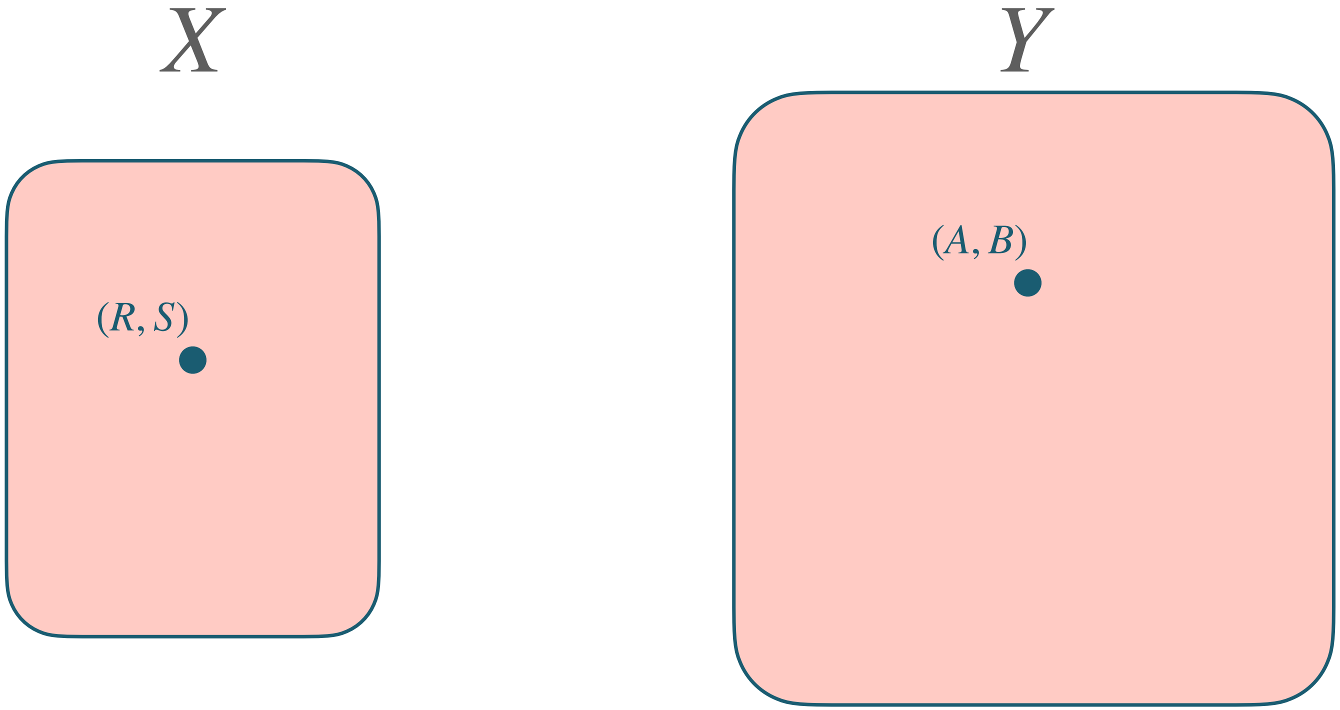
- **Proof**
  - Expansion property (sampler) of **query graph**

Query Graph



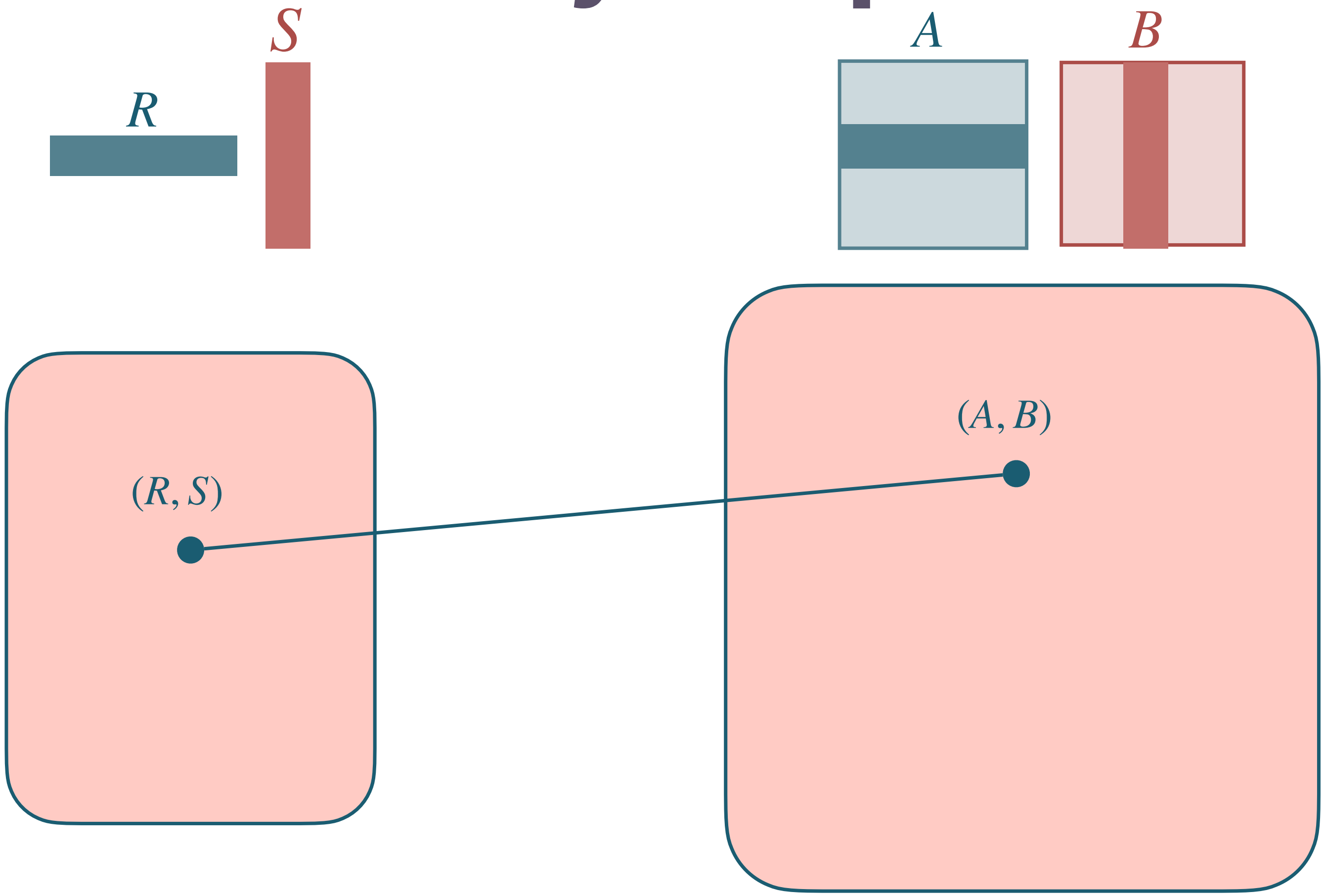
X = set of all of inputs (R, S)

Query Graph



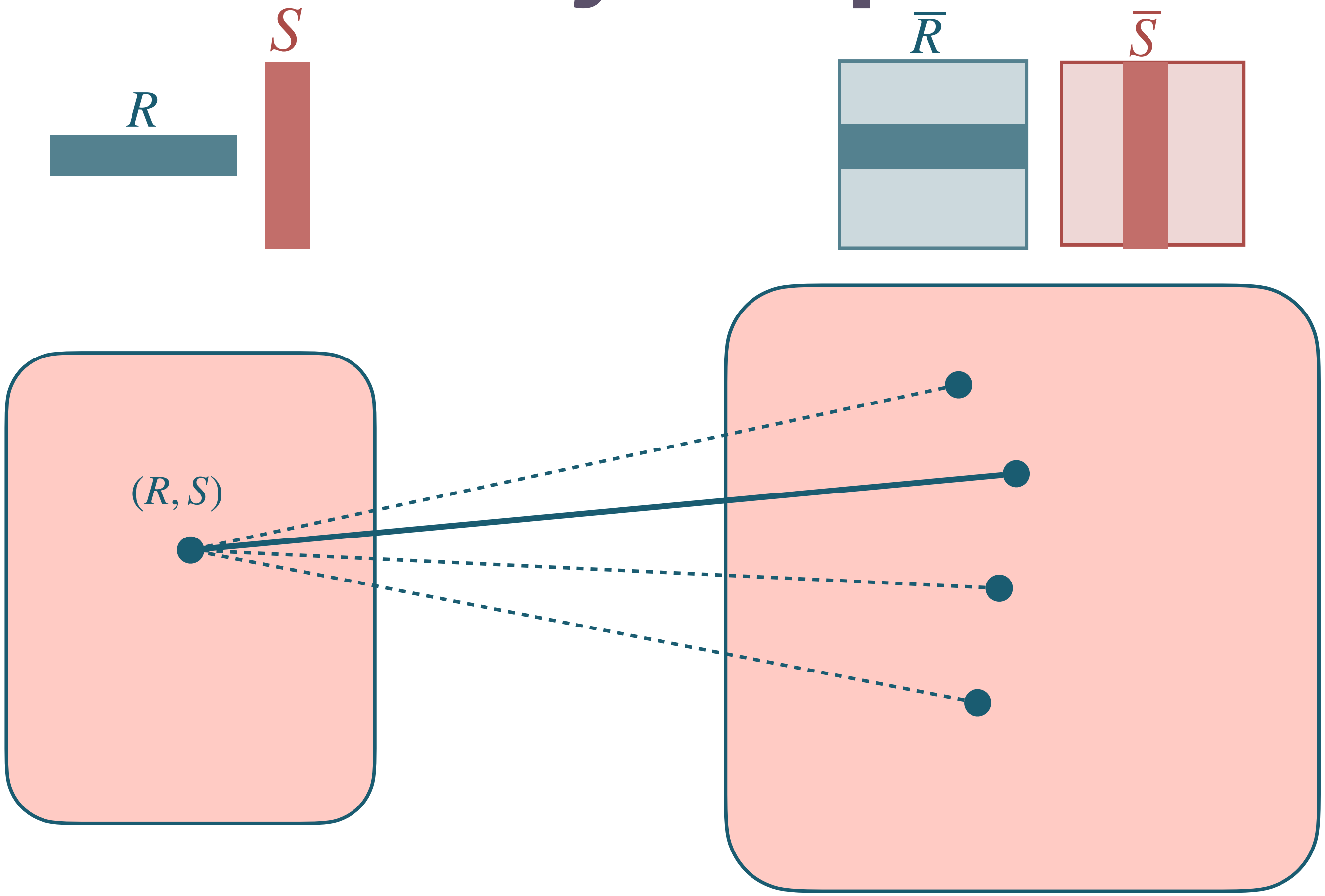
Y = set of all pairs (A, B) of $n \times n$ matrices

Query Graph



Edge weight = Pr[$\mathscr{R}(R, S)$ produces query (A, B)]

Query Graph

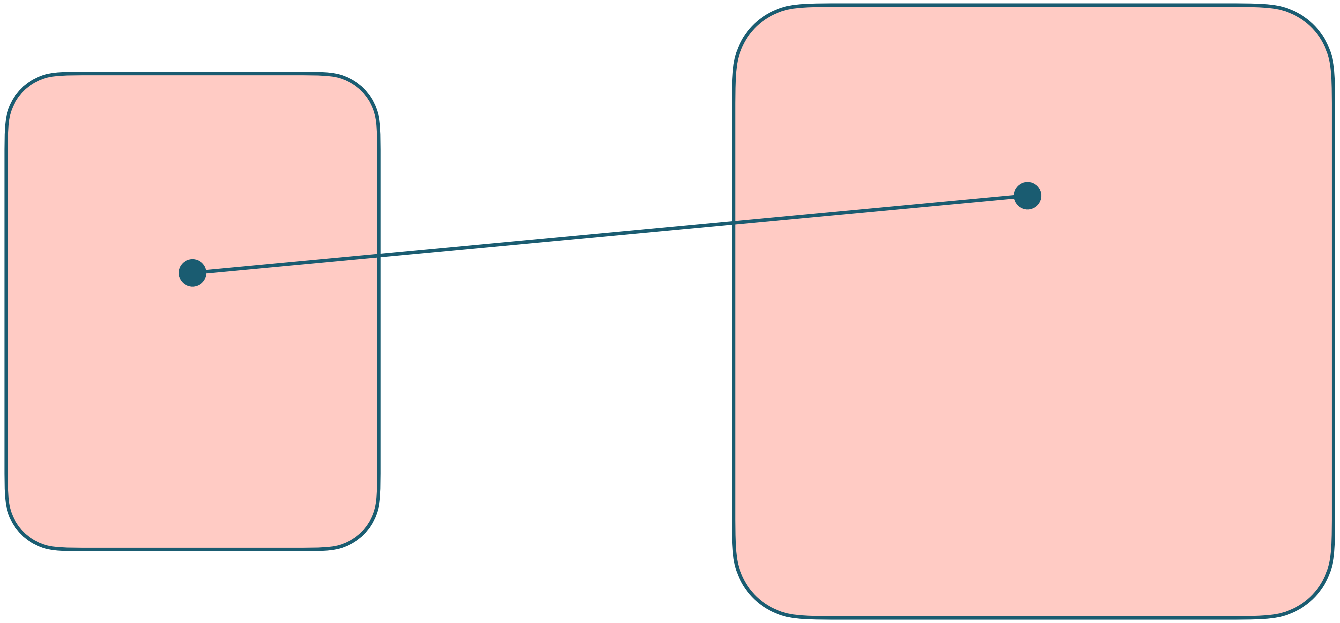


Query = random neighbor of (R, S)

Query Graph

Lemma (informal)

The query graph (X, Y, E) has an **expansion** property if $k \geq 100 \log(1/\epsilon)$



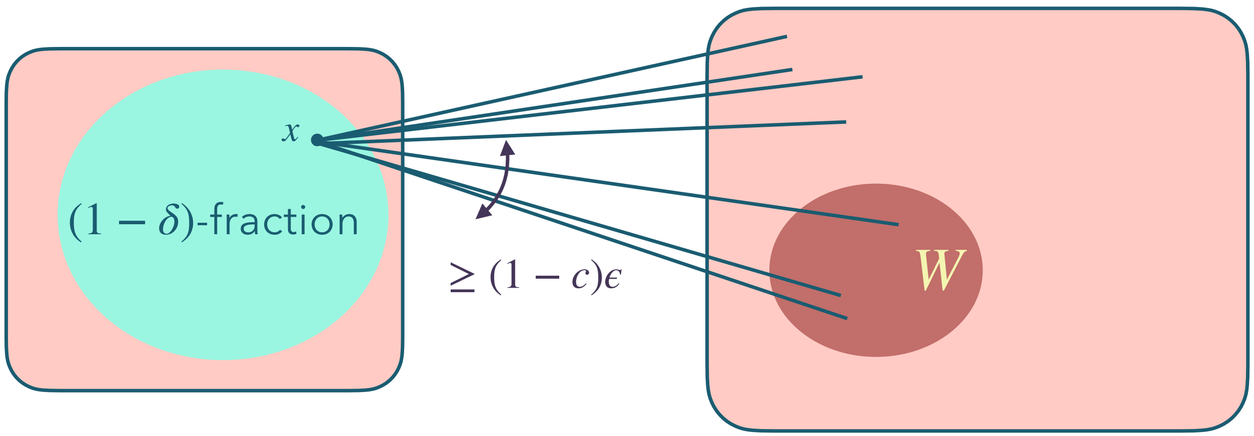
Sampler

Definition

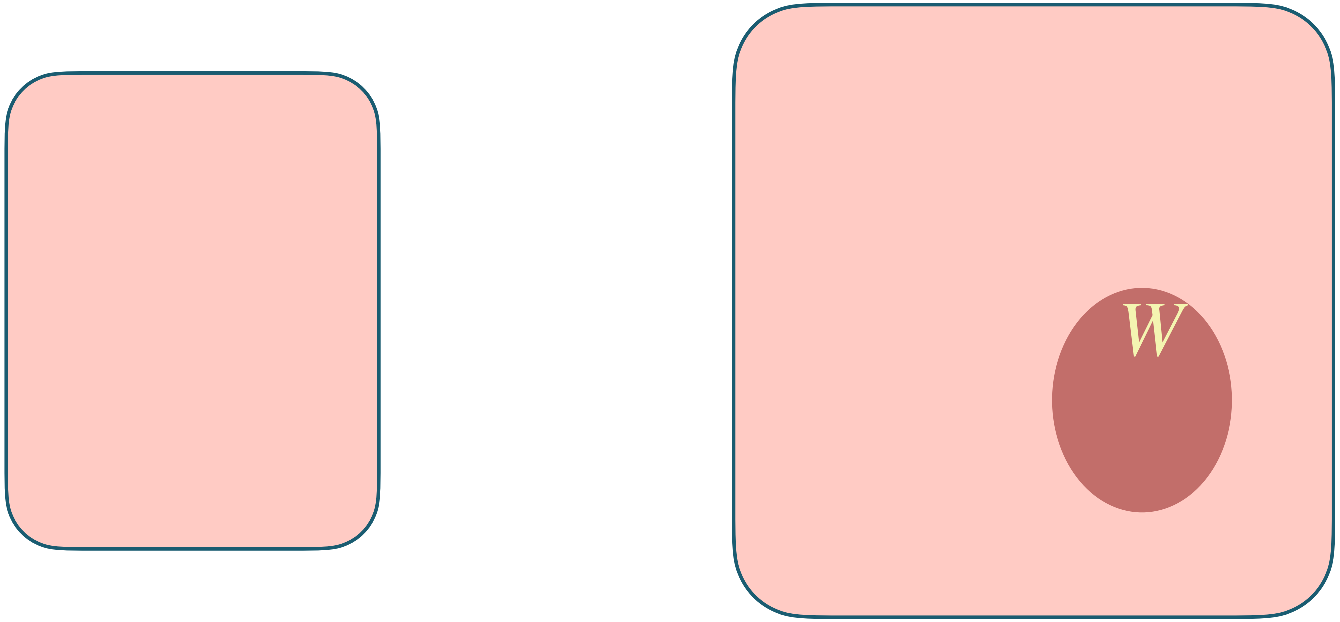
$Q = (X, Y, E)$ is (δ, c)-**sampler for density** ϵ if, for any $W \subseteq Y$ of $|W| \geq \epsilon |Y|$,

$$\Pr_{x \sim X}\left[\frac{|\Gamma(x) \cap W|}{|\Gamma(x)|} \geq (1-c)\epsilon\right] \geq 1 - \delta,$$

where $\Gamma(x) = \{\text{neighbors of } x\}$.

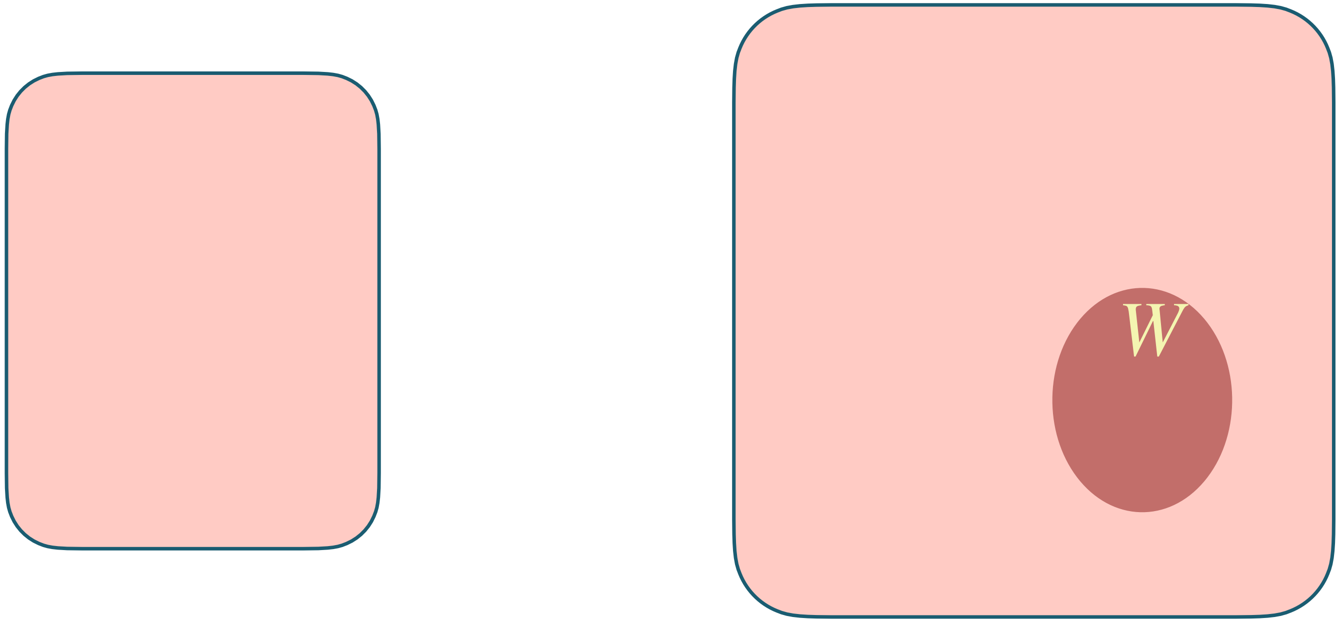


Query Graph



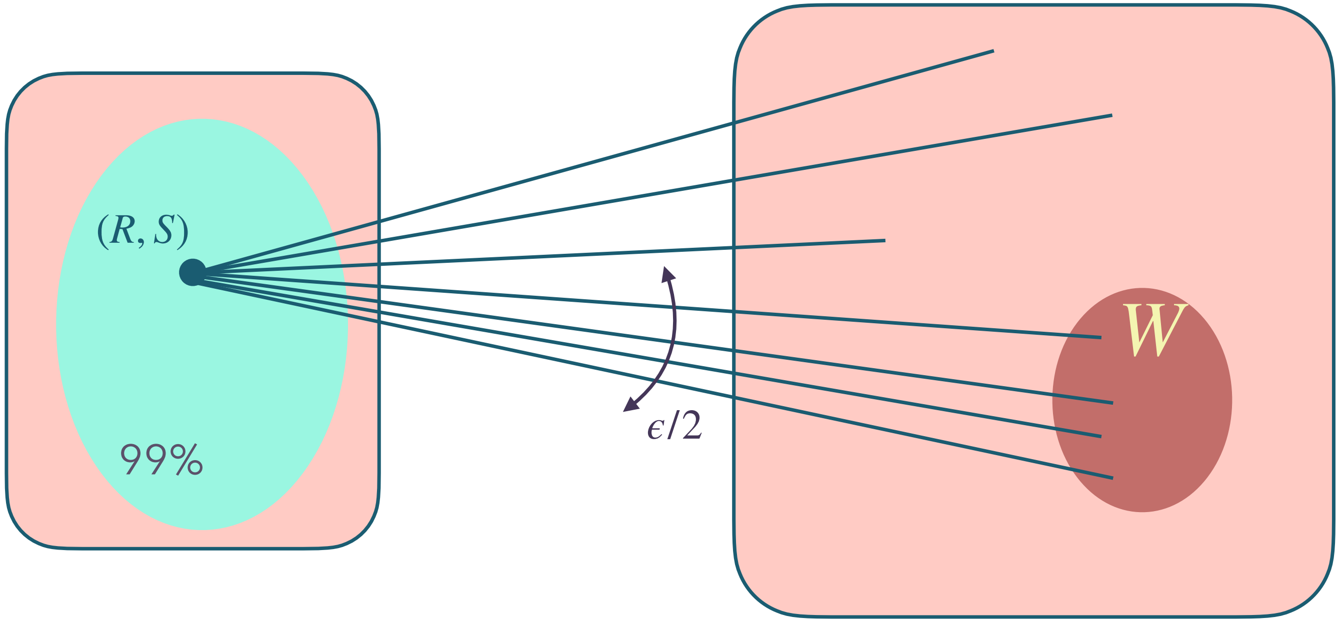
$$W = \{y \in Y \colon \mathscr{M}(y) \text{ succeeds}\}$$

Query Graph



W has density ϵ inside Y

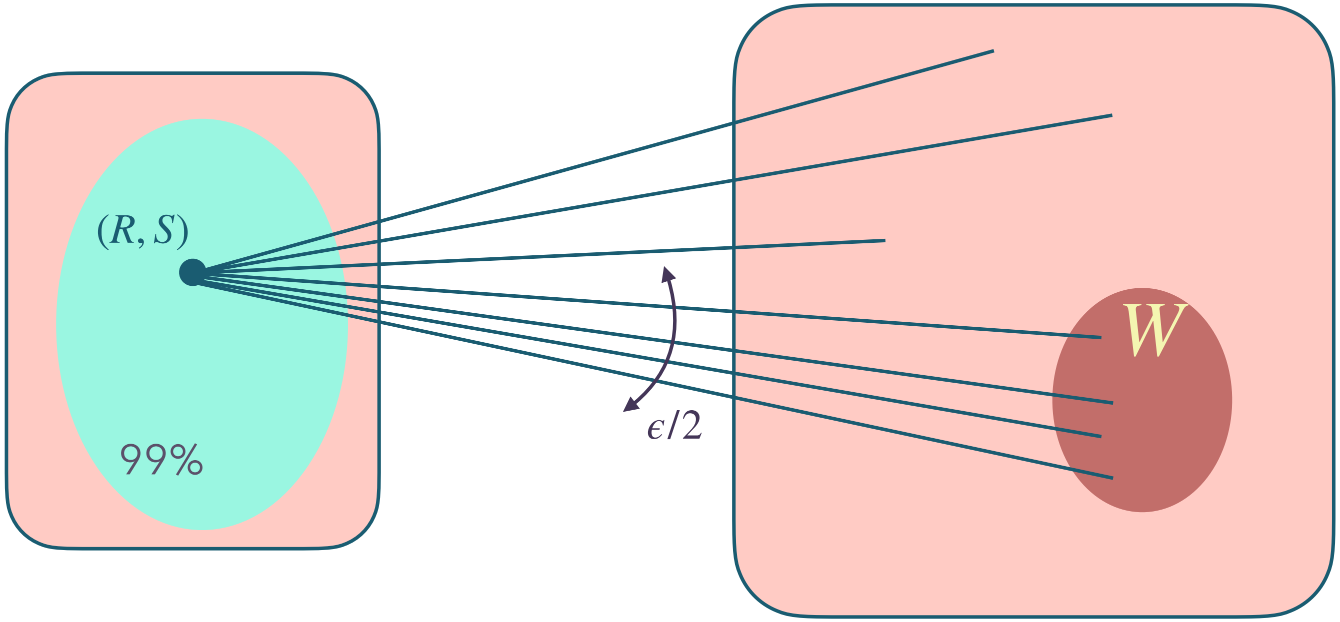
Query Graph



For $99\,\%$ of (R, S),

$\epsilon/2$-fraction of neighbors are in W

Query Graph



If we sample $O(1/\epsilon)$ random neighbors, one of them is in W

Query Graph

Lemma

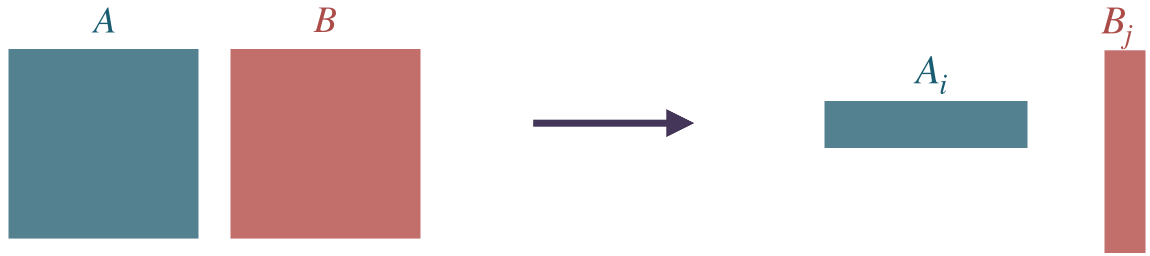
The query graph (X, Y, E) of MM is a (δ, c)-sampler for density ϵ if

$$k \geq \frac{8}{c^2\delta} \log\left(\frac{2}{c\epsilon}\right).$$

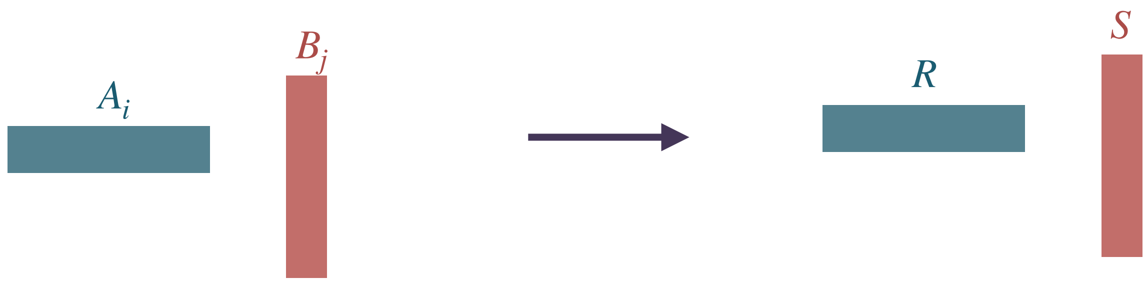
- In MM, we set $\delta = 0.99$ and $c = 1/2$
  - $k = O(\log(1/\epsilon))$ suffices

Proof Summary

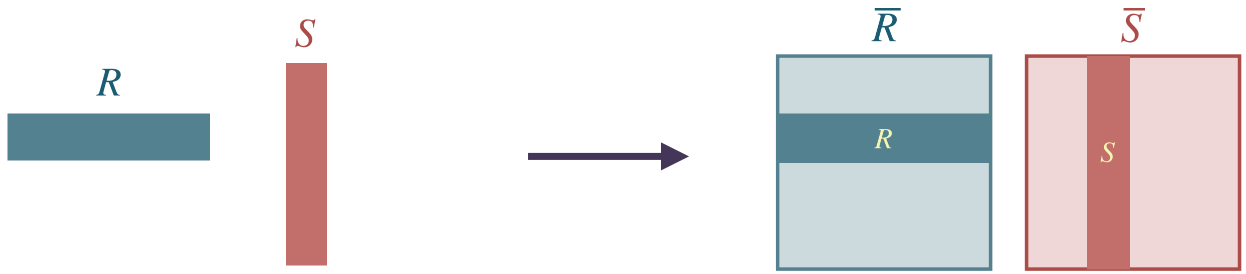
- **downward self-reduction**



- **random self-reduction**

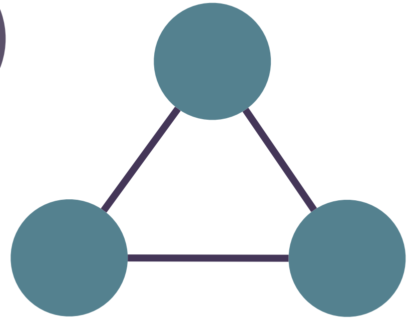


- **upward self-reduction**



Triangle Counting

Triangle Counting (TC)



- **Task: How many triangles (3-cycles) in a given graph?**
- **Input:** $G_{n,p}$ (for p const)

Theorem (Boix-Adserà, Brennan, Bresler, 2019)

If we can solve TC with success prob $1 - 1/\text{polylog}(n)$ in time $T(n)$, then we can solve TC in time $T(n) \cdot \text{polylog}(n)$ for any input.

- weak RSR
- strong RSR: **open**

weak hardness ← **worst-case hardness**

Triangle Counting (TC)

Theorem (this work)

$\exists T(n)$-time **error-less** algo for TC with success prob ϵ,

then $\exists \dfrac{T(n)\text{polylog}(n)}{\epsilon}$-time **nonuniform** randomized algo that solves TC for any input.

- **error-less algo**: output $\in \{\text{answer}, \perp\}$
  - never output a wrong value
- **nonuniform algo**: receives **advice string** α as additional input
  - α depends on **input size** n **& random seed**
- Proof: **Hardness self-amplification + BBB19**

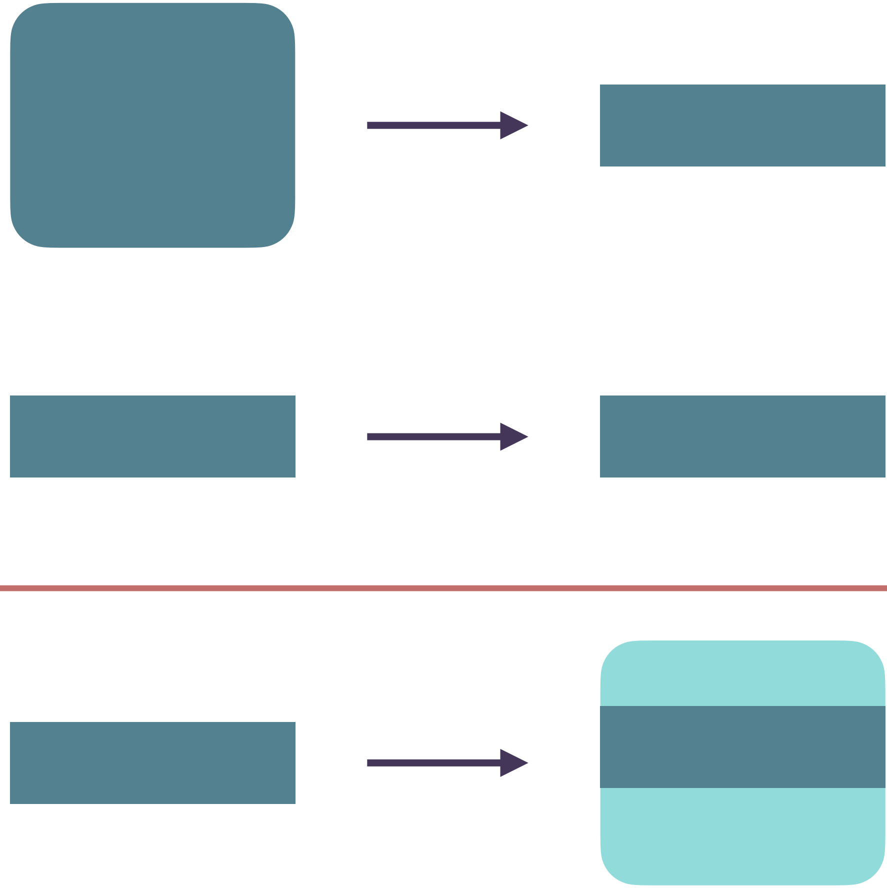
Related Work

- **Counting (over $G_{n,p}$)**
  - k-clique **[Boix-Adserà, Brennan, Bresler, 2019]**
  - general **[Dalirrooyfard, Lincoln, Williams, 2020]**
  - **low-error regime**
- **Counting Mod 2**
  - k-clique (low-error regime) **[Boix-Adserà, Brennan, Bresler, 2019], [Goldreich, 2020]**
  - triangle **(nonuniform, strong RSR)** **[Hirahara, S, 2022]**

We simplified & improved this reduction

Proof Summary

- **downward self-reduction**
  - n vertices → n/k vertices
- **random self-reduction**
  [Boix-Adserà, Brennan, Bresler, 2019]
- **upward self-reduction**
  - n/k vertices → n vertices
  - we use errorless + nonuniformity



Upward Reduction

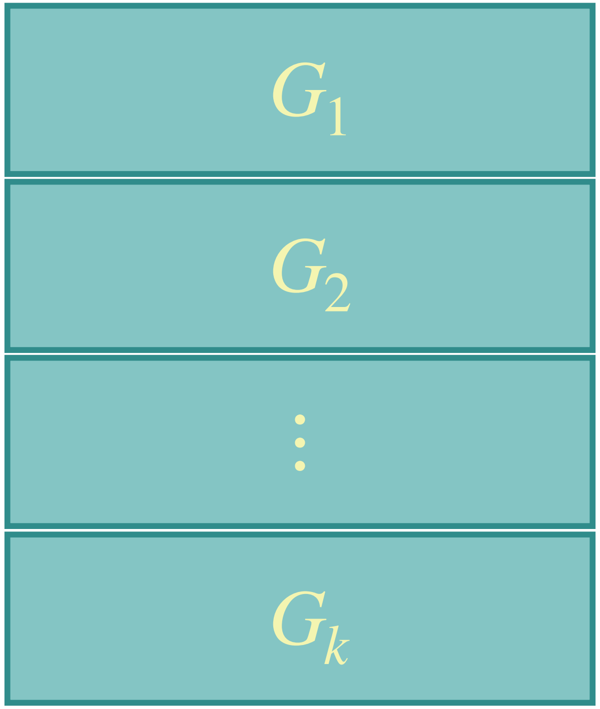
We have an algo $\mathscr{M}$ that solves TC with success prob ϵ over $G_{n,p}$

Goal: solve TC with success prob $1 - 1/\text{polylog}(n)$ over $G_{n/k,p}$

Upward Reduction

Input: $G \sim G_{n/k,p}$

Upward Reduction

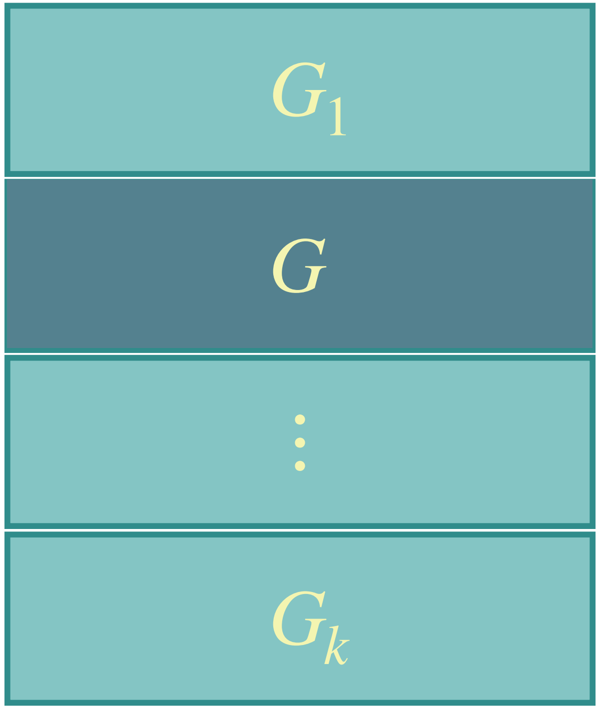


generate k graphs $G_1, \ldots, G_k \sim G_{n/k,p}$

Upward Reduction

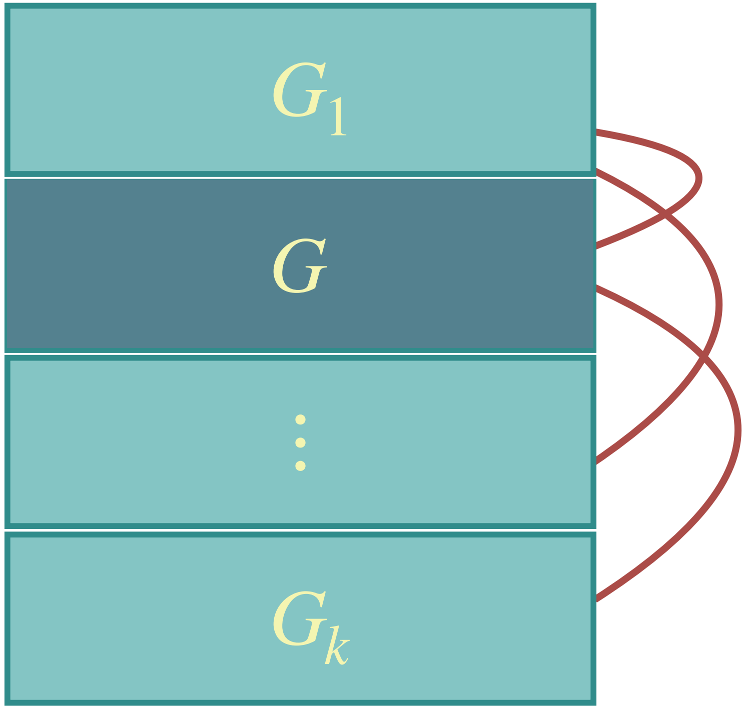
Select $i \sim [k]$

Upward Reduction



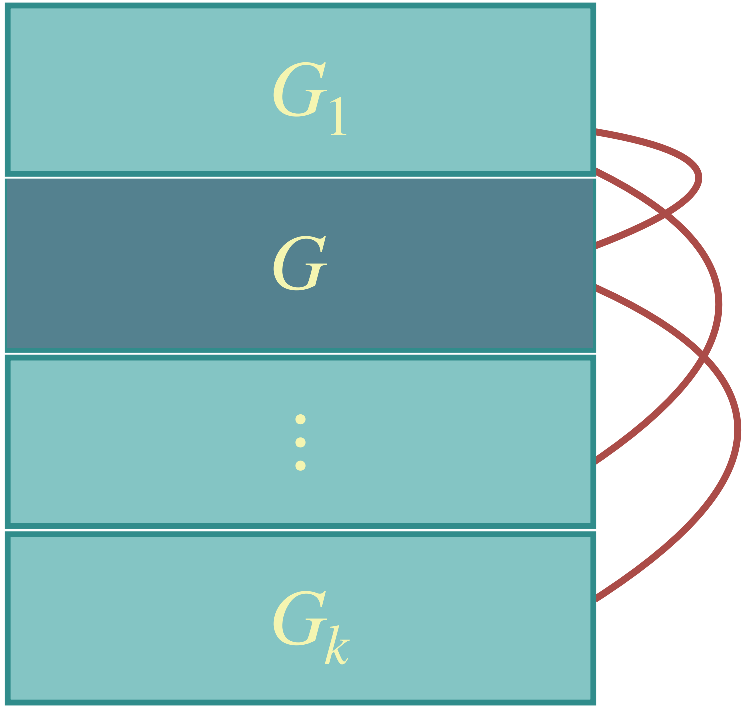
$$G_i \leftarrow G$$

Upward Reduction



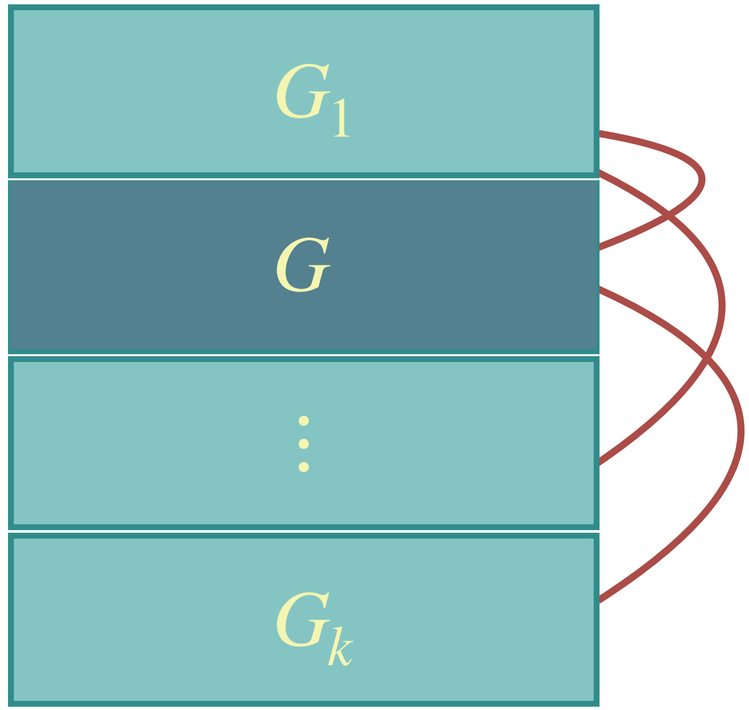
Add random edges between two groups
(with prob p)

Upward Reduction



Let $\overline{G}$ be the resulting graph
($\overline{G} \sim G_{n,p}$ since $G \sim G_{n/k,p}$)

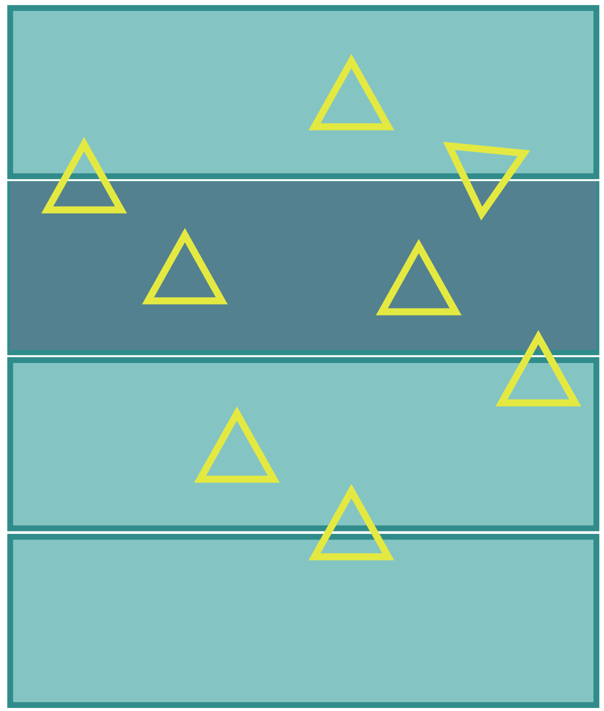
Upward Reduction



Run $\mathscr{M}(\overline{G})$.

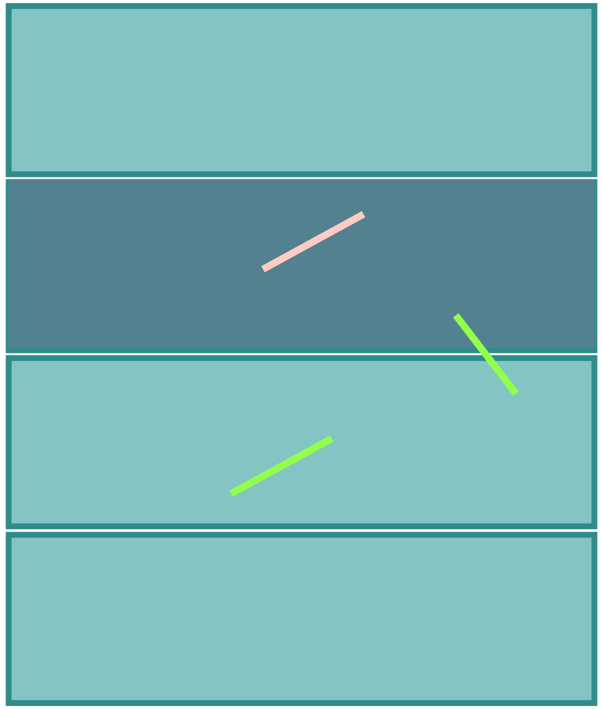
Upward Reduction

- **How to obtain $\#\text{Triangle}(G)$?**
  - In reduction, we have $\#\text{Triangle}(\overline{G})$ if $\mathscr{M}$ succeeds
  - this counts unnecessary triangles



Upward Reduction

- **How to obtain $\#\mathrm{Triangle}(G)$?**
  - In reduction, we have $\#\mathrm{Triangle}(\overline{G})$ if $\mathscr{M}$ succeeds
  - this counts unnecessary triangles
- **Two types of edges**
  - type 1: edges inside G
  - type 2: others



Upward Reduction

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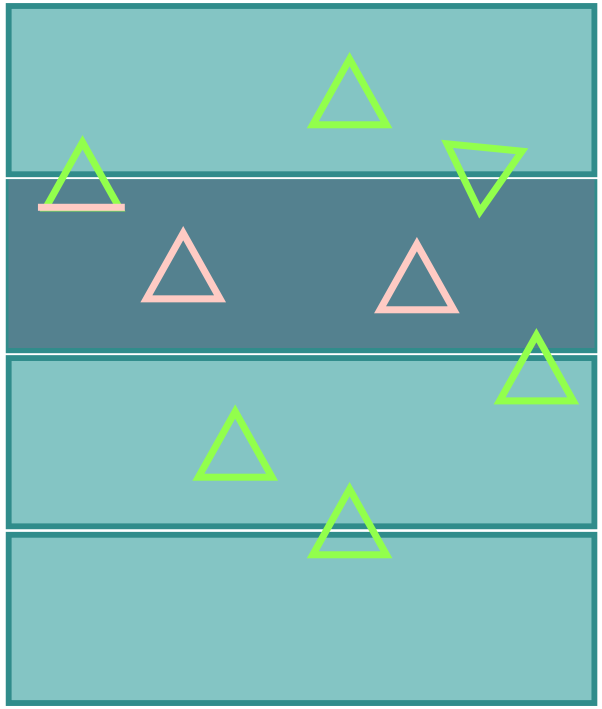
- **Three types of triangles**
  -
  -
  -



Upward Reduction

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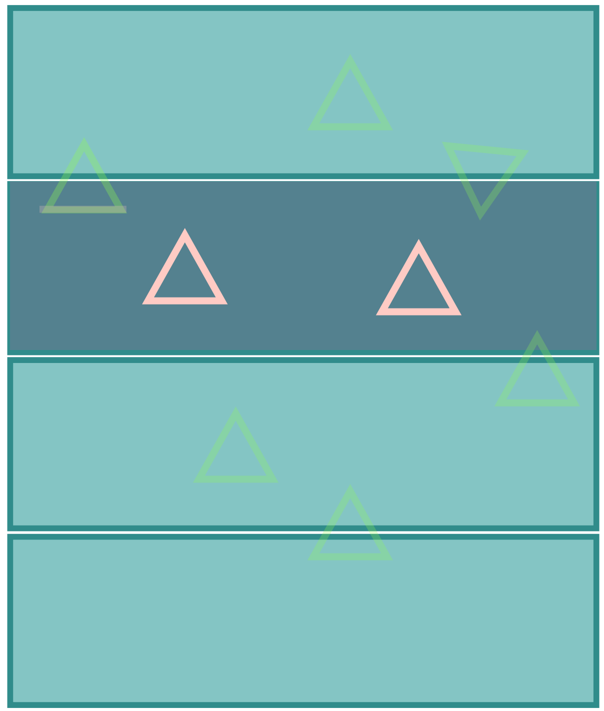
- **Three types of triangles**
  - type1 + type1 + type1 (we want to count)
  -
  -



Upward Reduction

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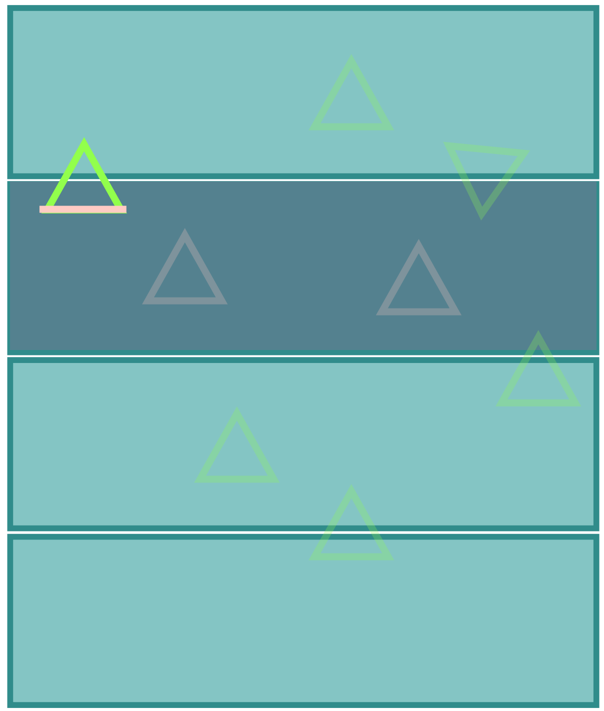
- **Three types of triangles**
  - type1 + type1 + type1
  - type1 + type2 + type2 (unnecessary)
  -



Upward Reduction

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- **Two types of edges**
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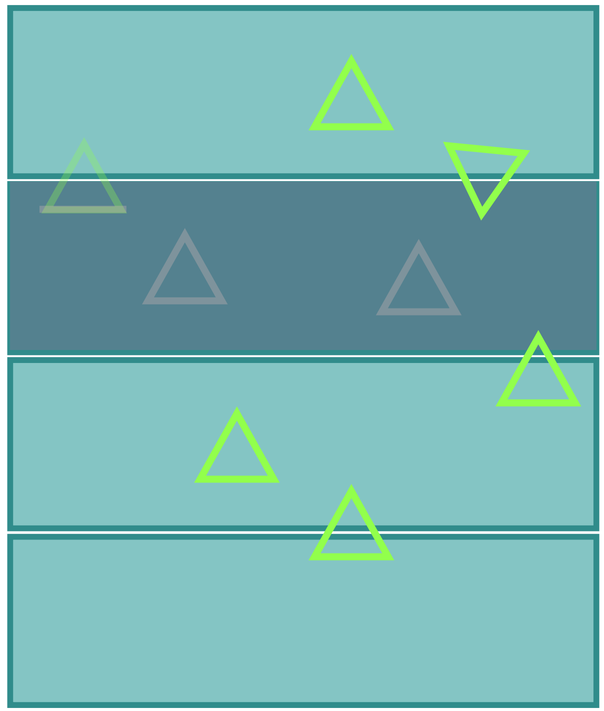
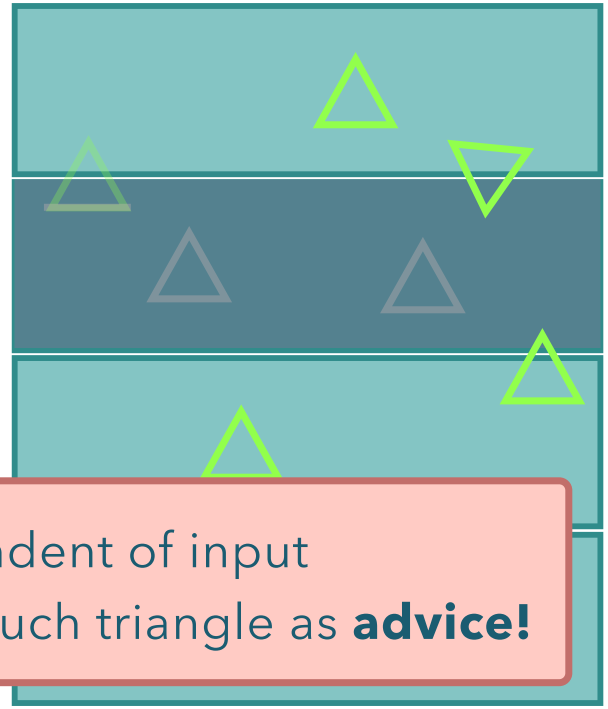
  - type 2: others
- **Three types of triangles**
  - type1 + type1 + type1
  - type1 + type2 + type2
  - type2 + type2 + type2 (unnecessary)



Upward Reduction

- **How to obtain $\#\mathrm{Triangle}(G)$?**
  - In reduction, we have $\#\mathrm{Triangle}(\overline{G})$ if $\mathscr{M}$ succeeds
  - this counts unnecessary triangles
- **Two types of edges**
  - type 1: edges inside G
  - type 2: others
- **Three types of triangles**
  - type1 + type1 + type1
  - type1 + type2 + type2
  - type2 + type2 + type2 (unnecessary)



all edges independent of input
we can give # of such triangle as **advice!**

Upward Reduction

- **How to obtain $\#\mathrm{Triangle}(G)$?**
  - In reduction, we have $\#\mathrm{Triangle}(\overline{G})$ if $\mathscr{M}$ succeeds
  - this counts unnecessary triangles
- **Two types of edges**
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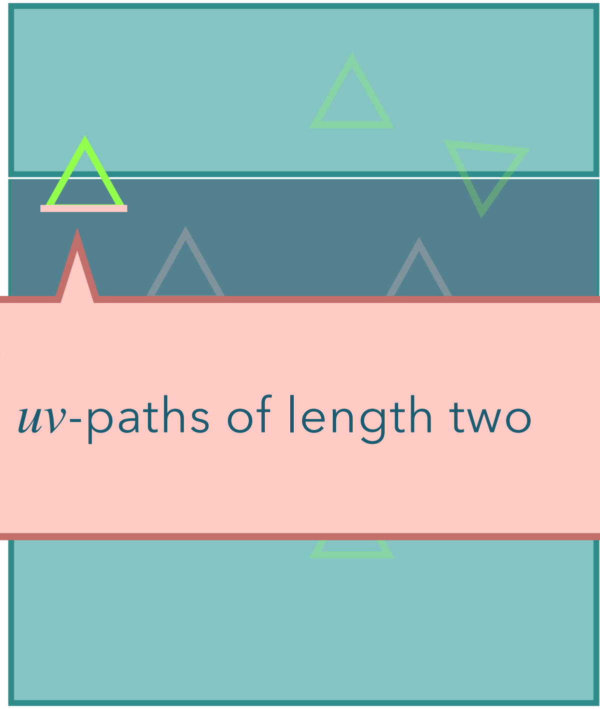

For each pink edge uv,
we can give # of green uv-paths of length two
as **advice**

Upward Reduction

- **How to obtain $\#\mathrm{Triangle}(G)$?**
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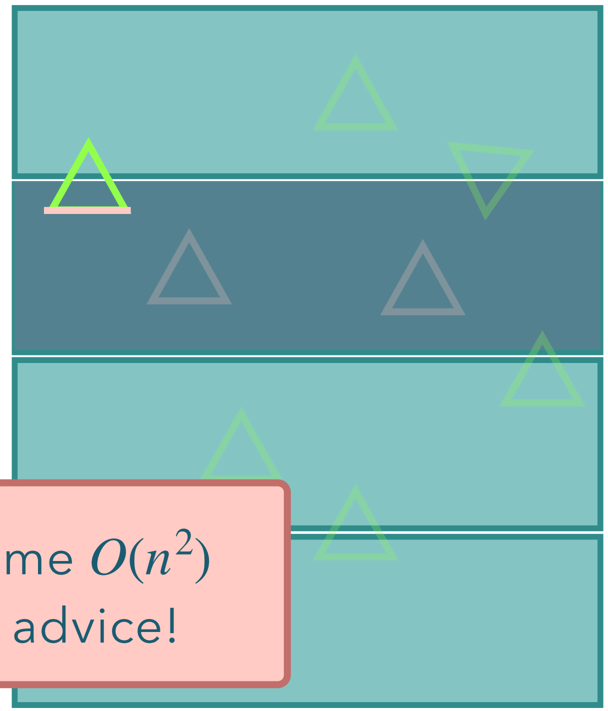
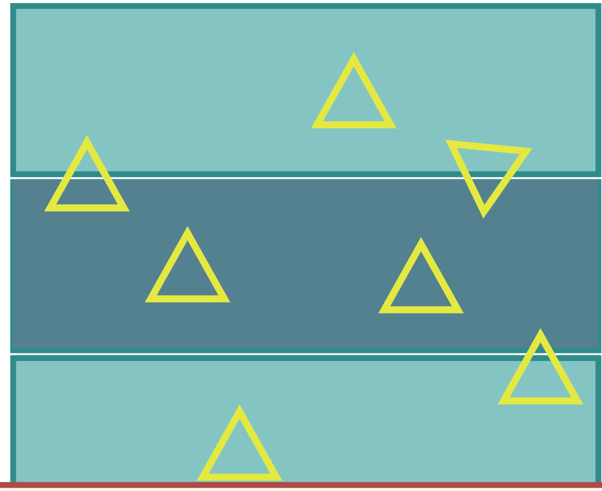


We can count in time $O(n^2)$ using nonuniform advice!

Upward Reduction

- **How to obtain $\#\mathrm{Triangle}(G)$?**
  - In reduction, we have $\#\mathrm{Triangle}(\overline{G})$ if $\mathscr{M}$ succeeds
  - this counts unnecessary triangles



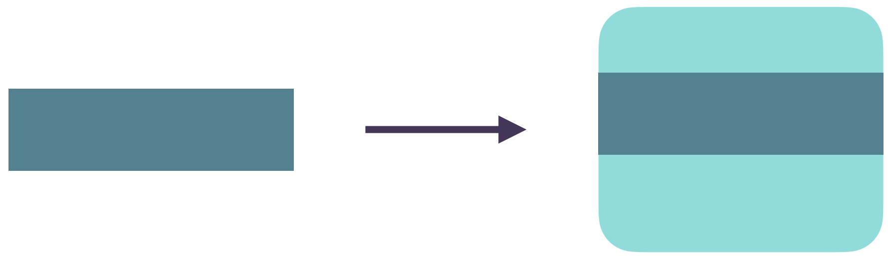
Lemma (informal)

We can compute $\#\mathrm{Triangle}(G)$ in nonuniform time $O(n^2)$ given $\#\mathrm{Triangle}(\overline{G})$.

Advice : $O(n^2 \log n)$ bits (# of green 2-paths)

Upward Self-Reduction

- **upward self-reduction**
  - n/k vertices $\to$ n vertices
  - we use errorless + nonuniformity



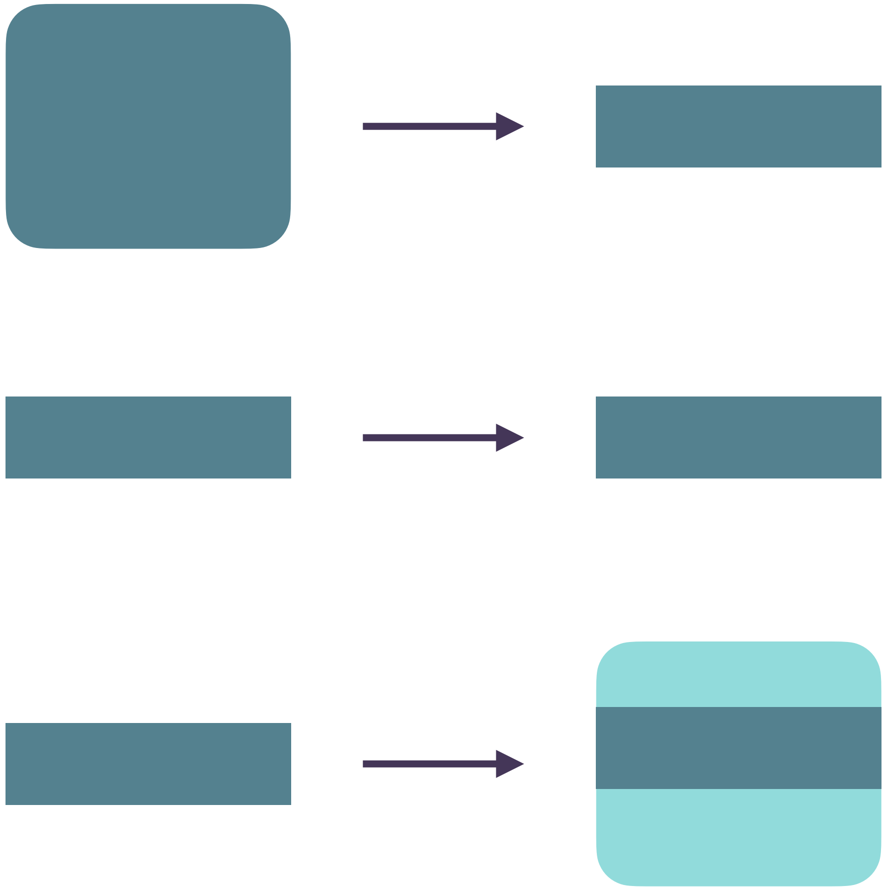
Lemma

Query graph is a sampler.

- **We can boost the success prob of $\mathscr{M}$ by repetition.**

Proof Summary

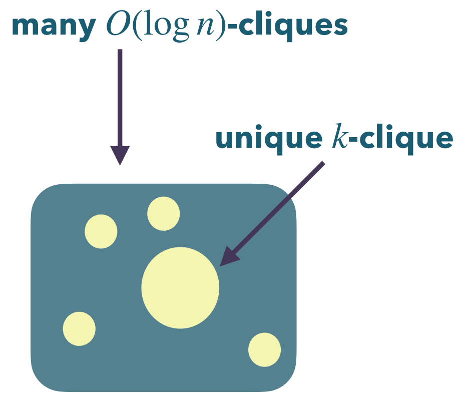
- **downward self-reduction**
  - n vertices → n/k vertices
- **random self-reduction**
  [Boix-Adserà, Brennan, Bresler, 2019]
- **upward self-reduction**
  - n/k vertices → n vertices
  - we use errorless + nonuniformity



Planted Clique

Random Graph with Planted Clique

- **Input: random k-clique + $G_{n,1/2}$ (Erdős–Rényi graph)**
  - Sample $G_{n,1/2}$
  - Randomly choose a set $C \subseteq V$ of k vertices
  - Make C a k-clique by adding edges
  - let $G_{n,1/2,k}$ be the resulting graph
- **Maximum clique of $G_{n,1/2}$** $\approx 2\log_2 n$
  - We assume $k \gg \log n$
  - Then, C is the unique k-clique (whp)



Search Planted Clique

Def (Search Planted Clique Problem) [Jerrum, 92][Kučera, 95]

Input : $G_{n,1/2,k}$

Output : any k-clique (not necessarily be the planted one)

- **If $k = \Omega(\sqrt{n})$, $\exists$ poly-time algo with success prob $1 - 2^{-n^{0.1}}$**
  - the larger k, the easier it it is to solve
- **open problem: poly-time algo for $\log n \ll k \ll \sqrt{n}$** [Alon, Krivelevich, Sudakov, 98] [Dekel, Gurel-Gurevich, Peres, 2014]

Decision Planted Clique

Def (Decision Planted Clique Problem)

Input : $G_{n,1/2,k}$ (with prob 1/2) **or** $G_{n,1/2}$ (with prob 1/2)

Output : "Yes" if the input contains a k-clique. "No" otherwise.

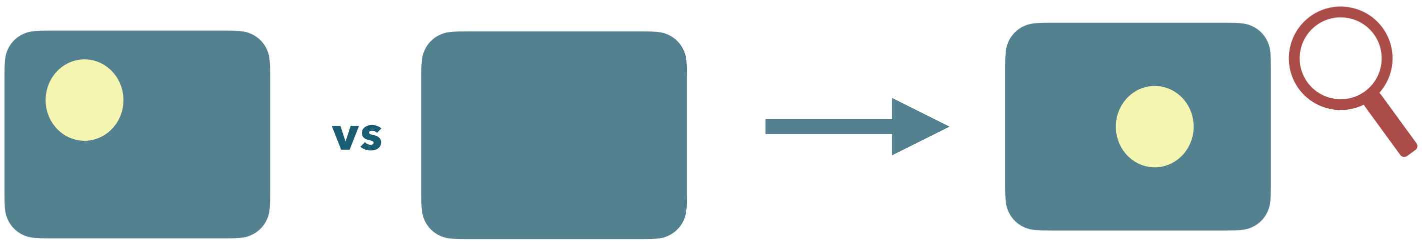
- $\mathscr{A}$ **has advantage** γ **if** $\Pr_G[\mathscr{A}(G)$ **is correct**$] \geq \dfrac{1+\gamma}{2}$
  - Random guess: $\gamma = 0$
  - Goal: $\gamma \approx 1$
- Algo for Search Planted Clique $\Rightarrow$ Algo for Decision Planted Clique
- **Does converse hold?**

Previous Work

Theorem (Alon, Andoni, Kaufman, Matulef, Rubinfeld, Xie, 2007).

If we can decide $G_{n,1/2,k}$ or $G_{n,1/2}$ with advantage $1 - 1/n^2$,

then, we can find a k-clique in $G_{n,1/2,k}$ with success prob $1 - 1/n$.



- **for low-error regime** 😔
  - reduction has n queries + union bound

Our Result

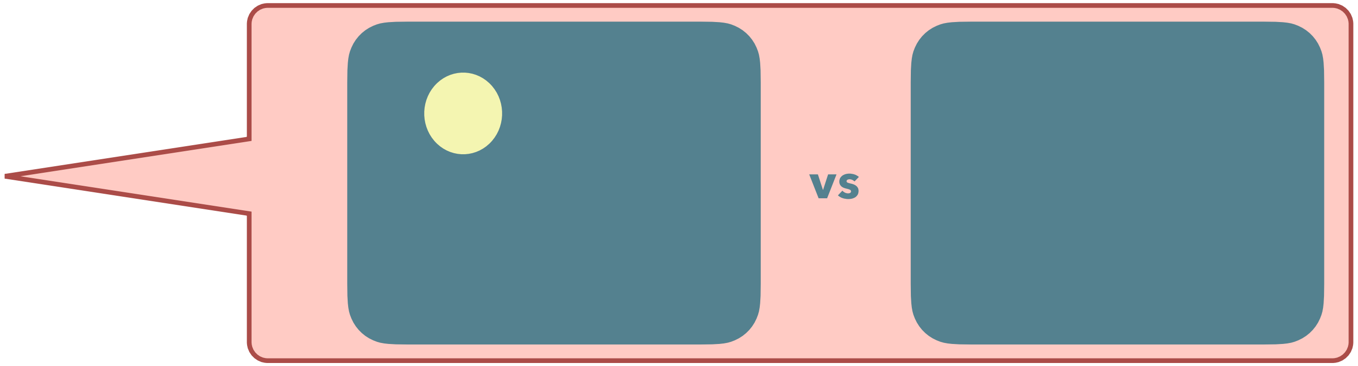
Theorem.

If we can decide $G_{N,1/2,k}$ or $G_{N,1/2}$ with advantage $\epsilon(N) \geq N^{-1/2+c}$, then, we can find a k-clique in $G_{n,1/2,k}$ with success prob $1 - 1/n$, where $N = n^{O(1/c)}$.

- high-error regime!
- Blow-up in instance size 😔

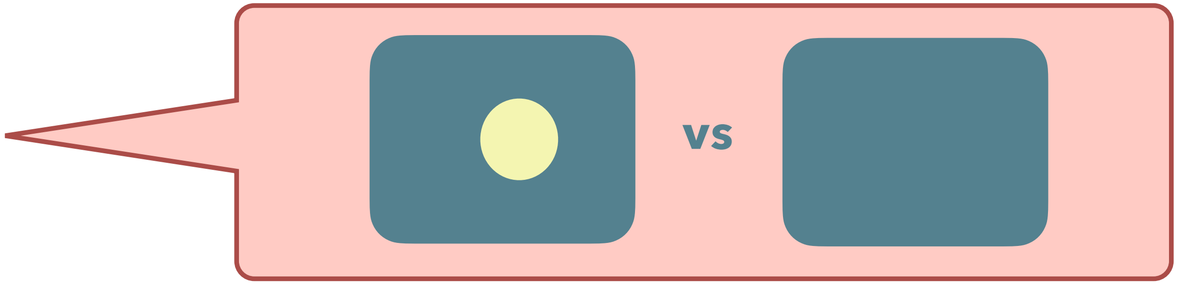
Proof Outline

decision algo with adv ϵ



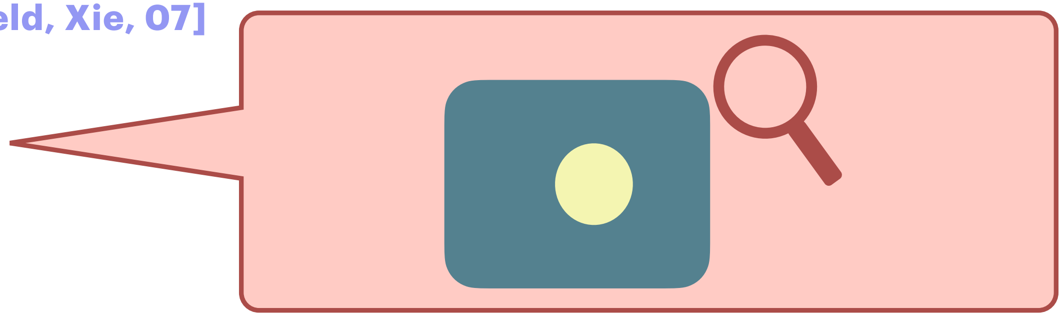
hardness amplification
polynomial blow-up in n

decision algo with adv $1 - 1/n^2$



Search-to-Decision by
[Alon, Andoni, Kaufman, Matulef, Rubinfeld, Xie, 07]

search algo with success prob $1 - 1/n$



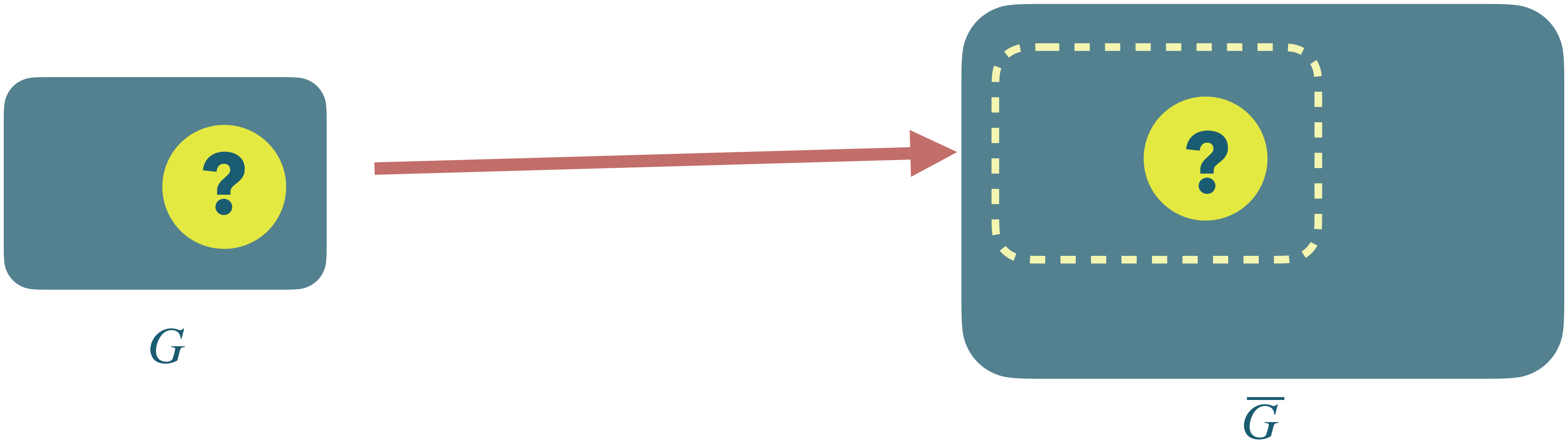
Our Reduction

- For simplicity we focus on Search Planted Clique
- $\mathscr{A}$: algo with success prob ϵ
- G : input (chosen from $G_{n,1/2,k}$)

G

Our Reduction

- **For $N = \mathrm{poly}(n)$, randomly embed G into $G_{N,1/2}$. Let $\overline{G}$ be the resulting graph.**
  - Let $\mathscr{R}^{\mathscr{A}}(G)$ be the randomized reduction that outputs $\mathscr{A}(\overline{G})$
- **Repeat $\mathscr{R}^{\mathscr{A}}(G)$ until $\mathscr{A}(\overline{G})$ outputs a k-clique in G**
- **$\overline{G}$ contains a unique k-clique since $k \gg \log N$**



Analysis

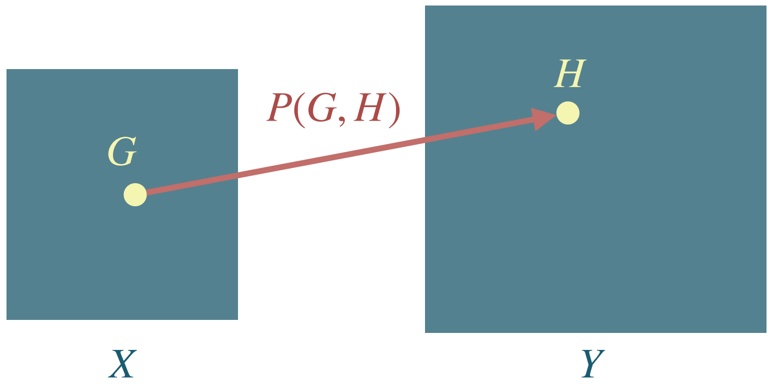
Def (Query Graph)

The query graph is the edge-weighted bipartite graph $Q = (X, Y, P)$ defined by

$X =$ **set of all n-vertex graph having a k-clique**

$Y =$ **set of all N-vertex graph having a k-clique**

$P(G, H) = \Pr[\mathscr{R}^{\mathscr{A}}(G)$ produces query $H]$



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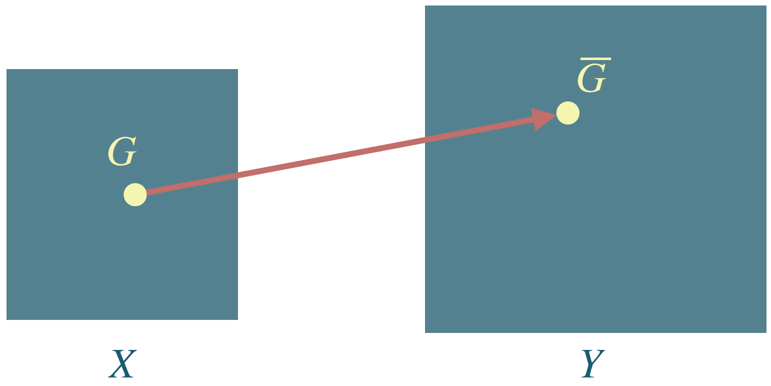


$\overline{G}$ is a random neighbor of G

Analysis

Def (Query Graph)

The **query graph** is the edge-weighted bipartite graph $Q = (X, Y, P)$ defined by

$X =$ set of all n-vertex graph having a k-clique

$Y =$ set of all N-vertex graph having a k-clique

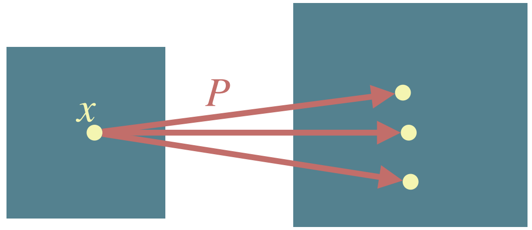
$P(G, H) = \Pr[\mathscr{R}^{\mathscr{A}}(G) \text{ produces query } H]$

Lemma

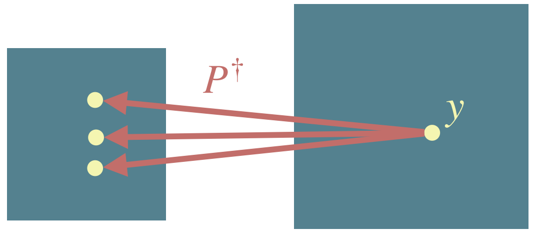
The query graph Q is a (δ, c)-sampler for density ϵ if $N \geq \dfrac{n}{c^2 \delta \epsilon}$

Sampler and Expander

- **Let** $P = [0,1]^{X \times Y}$ **be** $P(x, y) = \dfrac{1}{|N(x)|}$
  - $P(x, \cdot\,) =$ upward random walk



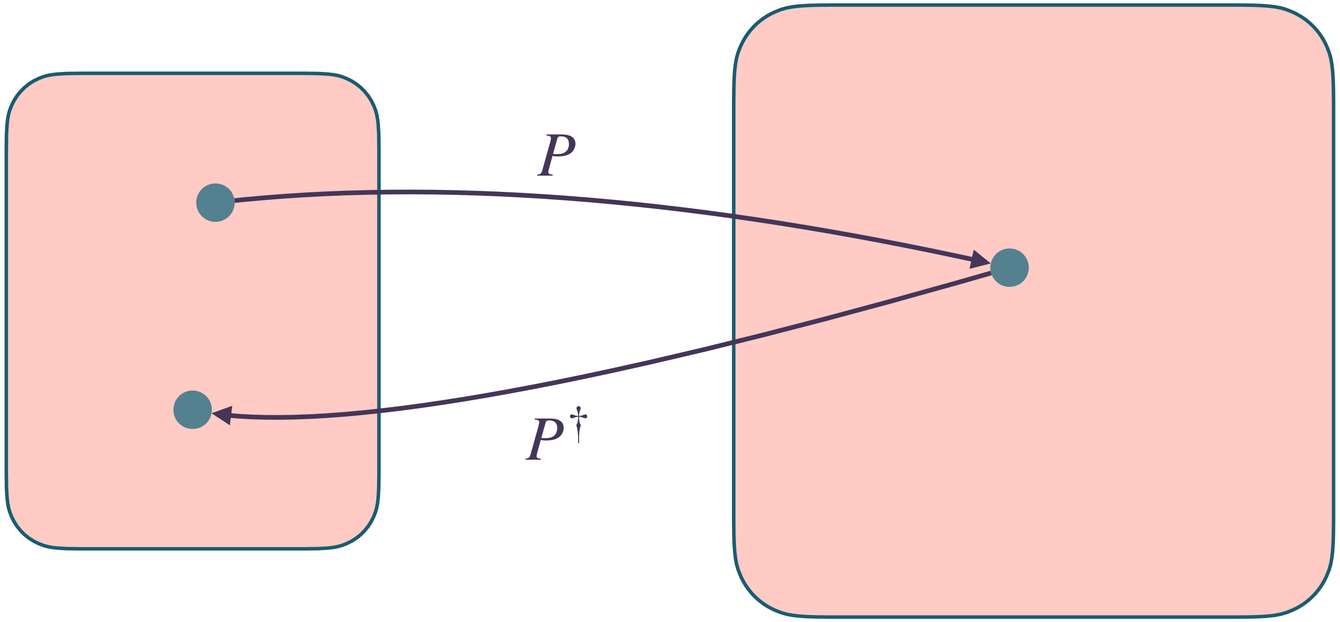
- **Let** $P^\dagger \in [0,1]^{Y \times X}$ **be** $P^\dagger(y, x) = \dfrac{1}{|N(y)|}$
  - $P^\dagger(y, \cdot\,) =$ downward random walk



Lemma (informal)

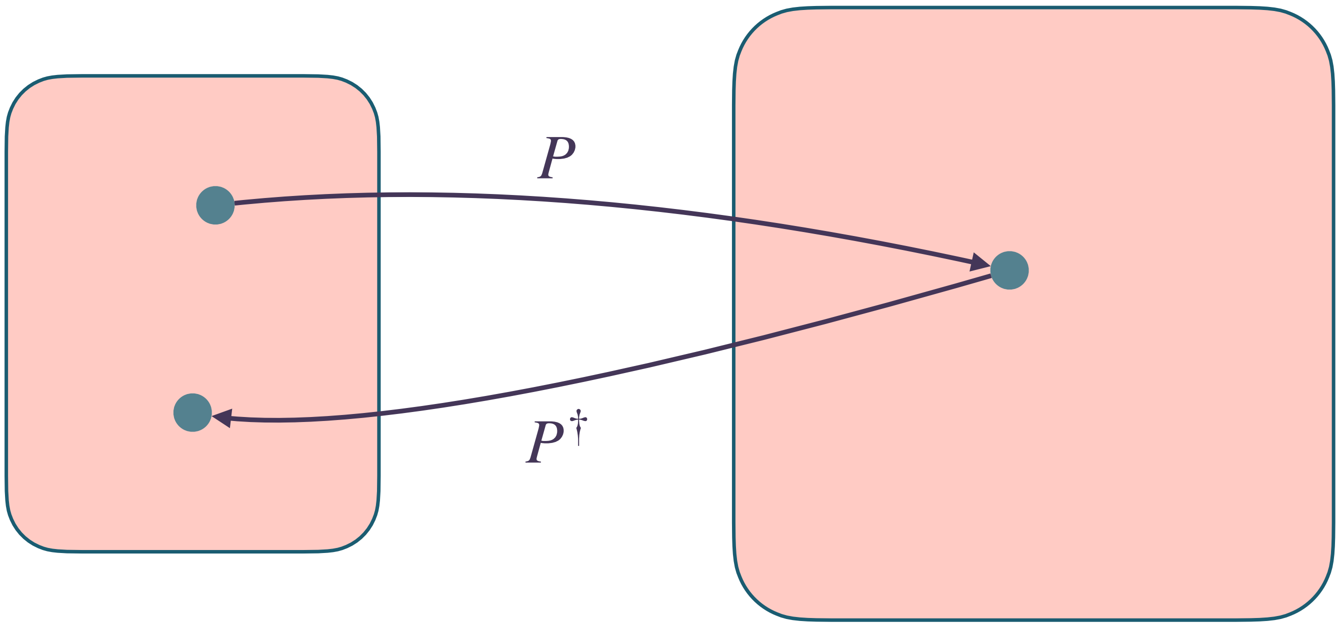
If $\lambda_2(PP^\dagger)$ is small, then Q is a sampler

Up-Down Walk



To bound $\lambda_2(PP^\dagger)$, we need **rapid mixing** of RW according to $PP^\dagger$

Up-Down Walk



This can be done by **coupling technique** of Markov chain

Why $\epsilon(N) \gg N^{-1/2}$?

Theorem.

If we can decide $G_{N,1/2,k}$ or $G_{N,1/2}$ with advantage $\epsilon(N) \geq N^{-1/2+c}$, then, we can find a k-clique in $G_{n,1/2,k}$ with success prob $1 - 1/n$, where $N = n^{O(1/c)}$.

- **Blow-up in instance size**
  - For MM and TC, we used downward self-reduction to preserve instance size.
- **For decision PC problem, we need a $(\delta, \epsilon/2)$-sampler for density $1/2 + \epsilon$**
  - Query graph is sampler if $N \geq \Theta\left(\frac{n}{\delta\epsilon^2}\right)$
- Here, $\epsilon = \epsilon(N)$ and thus, $\epsilon^2 \geq n/N = N^{-1+c}$ if $N = n^c$

Why $\epsilon(N) \gg N^{-1/2}$?

Theorem.

If we can decide $G_{N,1/2,k}$ or $G_{N,1/2}$ with advantage $\epsilon(N) \geq N^{-1/2+c}$, then, we can find a k-clique in $G_{n,1/2,k}$ with success prob $1 - 1/n$, where $N = n^{O(1/c)}$.

- **Blow-up in instance size**
  - For MM and TC, we used downward self-reduction to preserve instance size.
- **For decision PC problem, we need a $(\delta, \epsilon/2)$-sampler for density $1/2 + \epsilon$**

Open Question.

Can we improve the dependency of N on $1/\epsilon$?

(in particular, we are interested in $\log(1/\epsilon)$)

Conclusion

- query graph is sampler $\Rightarrow$ hardness amplification
    - Matrix Multiplication
    - Online Matrix-Vector Multiplication
    - Triangle Counting
    - Planted Clique
- **Reduction**: Downward/Upward/Random Self-Reduction + Sampler
- **Further Application**: other "planted" problems (e.g., planted k-SUM)
- **Open**:
    - Improve the blow-up of $N = \mathrm{poly}(n)$
    - Uniform reduction for triangle
    - General subgraph counting