A ROBUST VERSION OF HEGEDUS'S LEMMA

WITH APPLICATIONS

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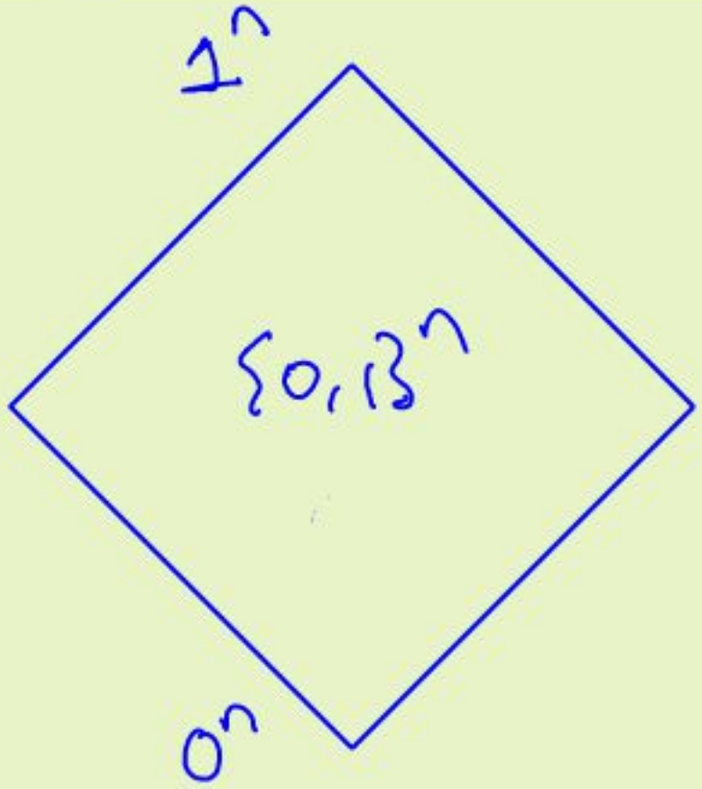
(DEPT. OF MATH., IIT BOMBAY)

OXFORD - WARWICK COMPLEXITY MEETINGS

Polynomials over the hypercube



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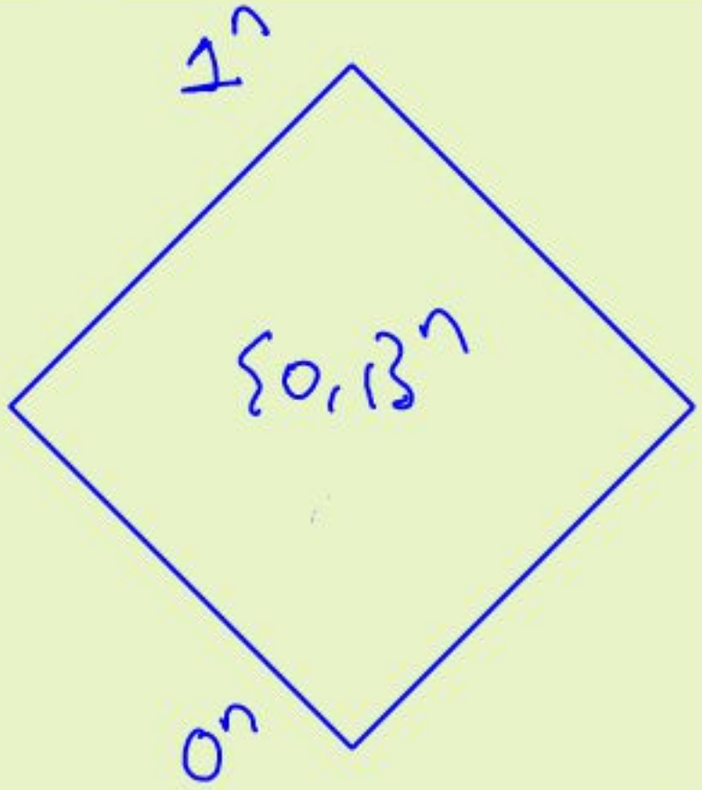

Combinatorial/algebraic properties of low degree polynomials over $\{0,1\}^n$

$R \in \mathbb{F}[x_1, \ldots, x_n]$

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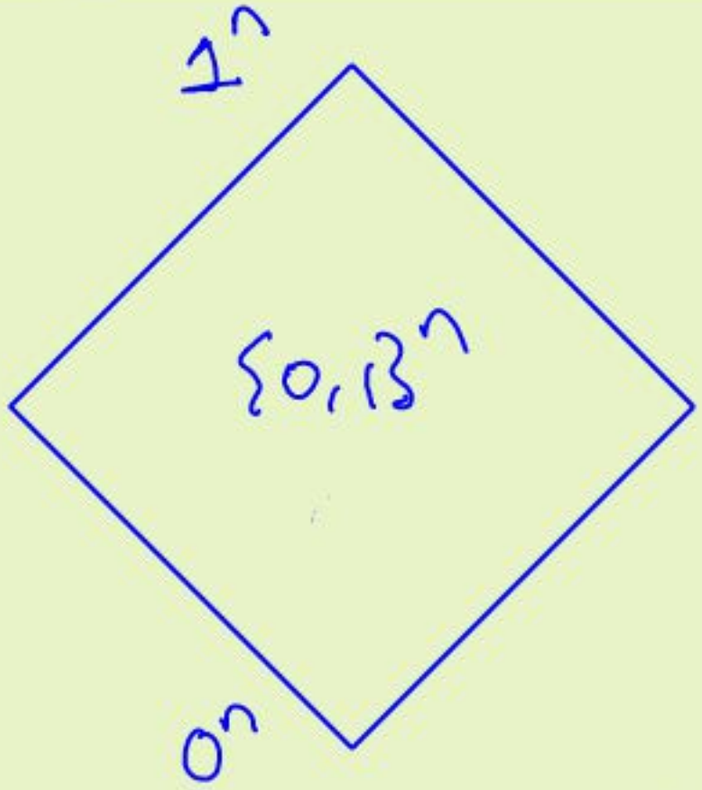

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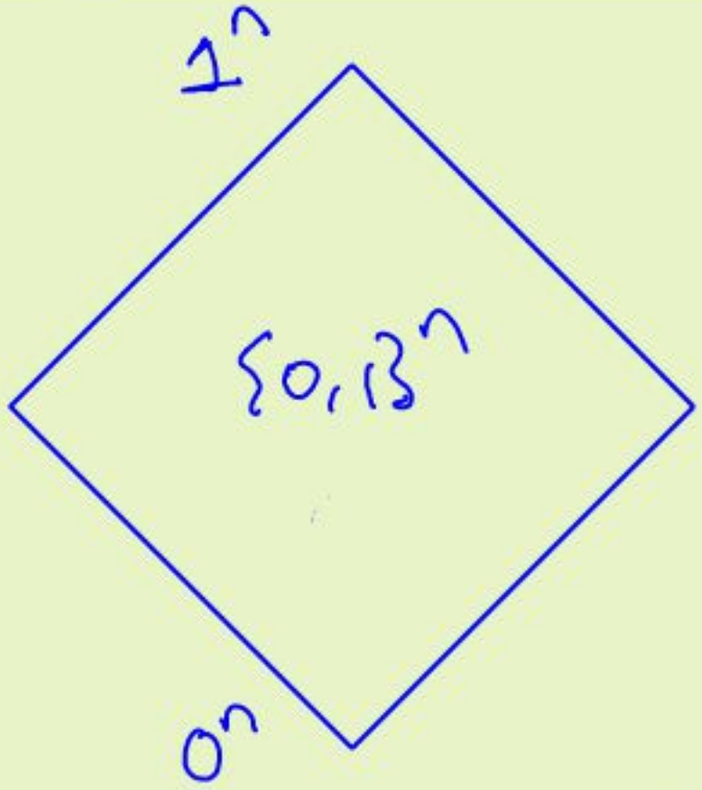

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1. Complexity theory
   - → Circuit complexity [Razborov '87]
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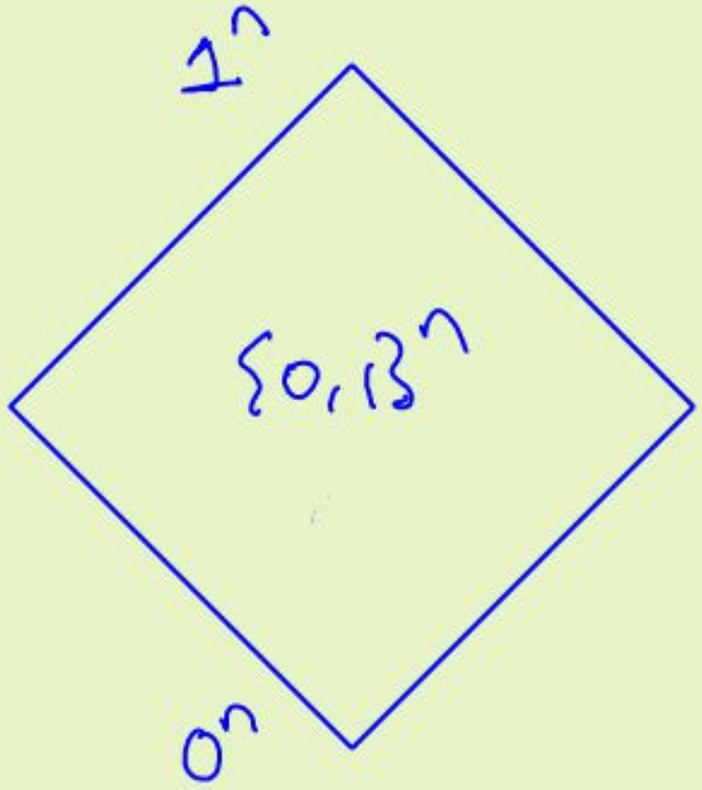

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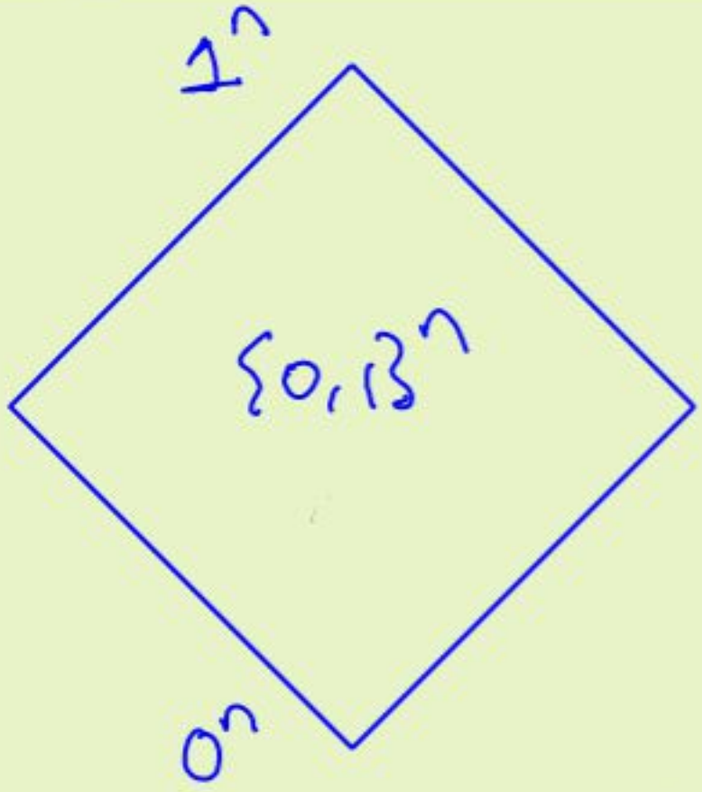

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3. Combinatorial algos [Williams '14]

Hegedüs's lemma

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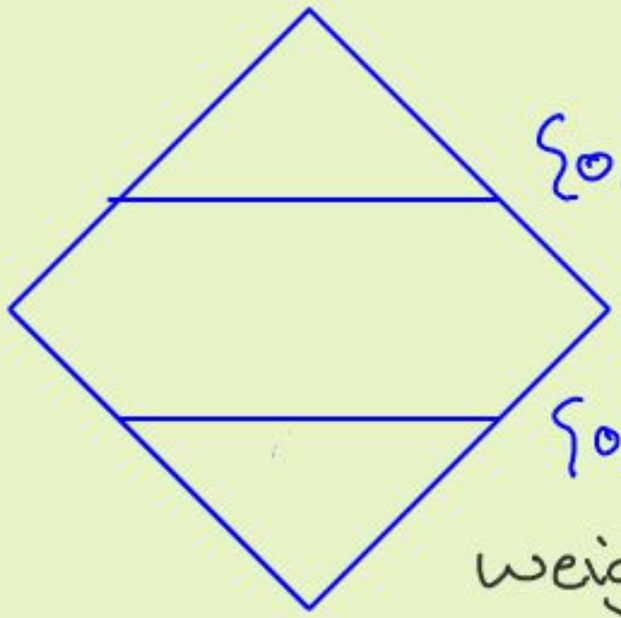


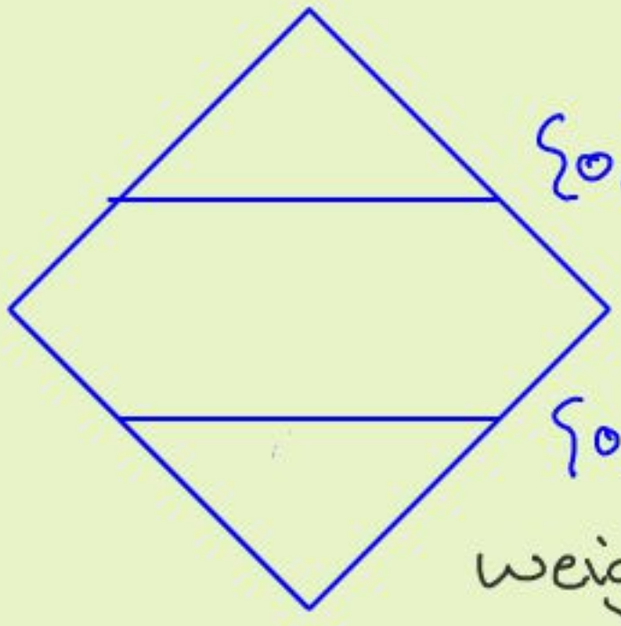
$\{0,1\}^n_{k+q}$

$\{0,1\}^n_k$

weight k

Hegedüs's lemma

$\mathbb{F}[x_1, \dots, x_n]$



Q: lowest deg. $R(x_1, \dots, x_n)$ s.t.

(I) $|a| = k \Rightarrow R(a) = 0$

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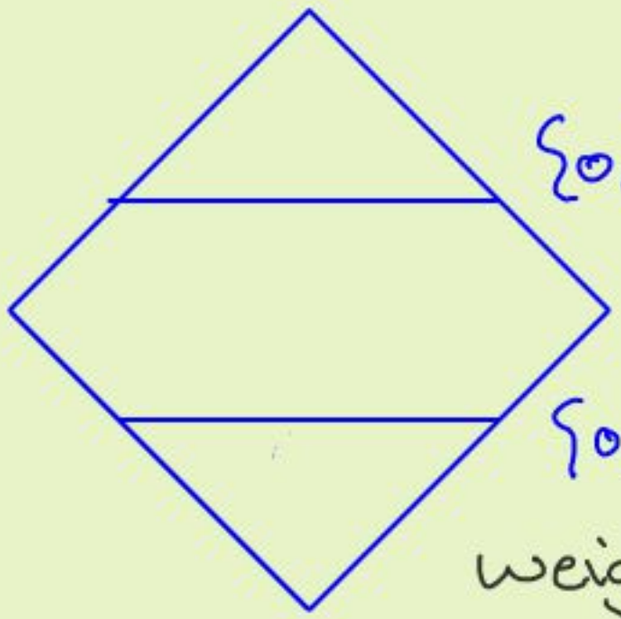

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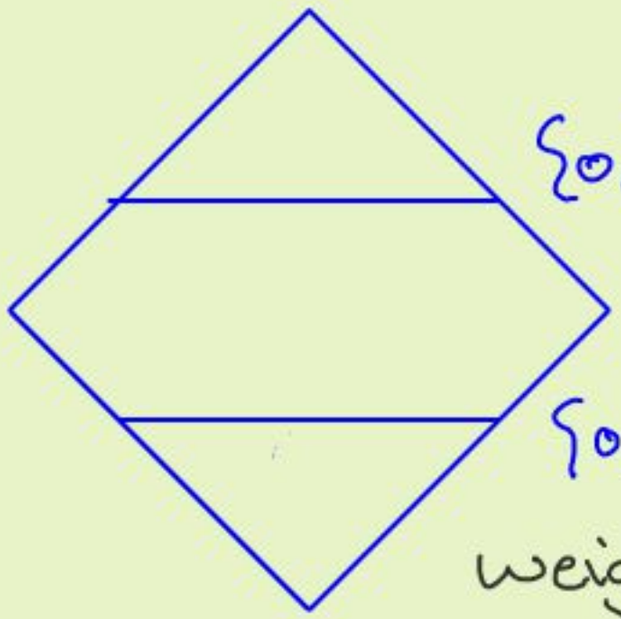

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$\mathbb{F} = \mathbb{R}$: $R(x_1, \dots, x_n) = \left(\sum_i x_i - k\right)$

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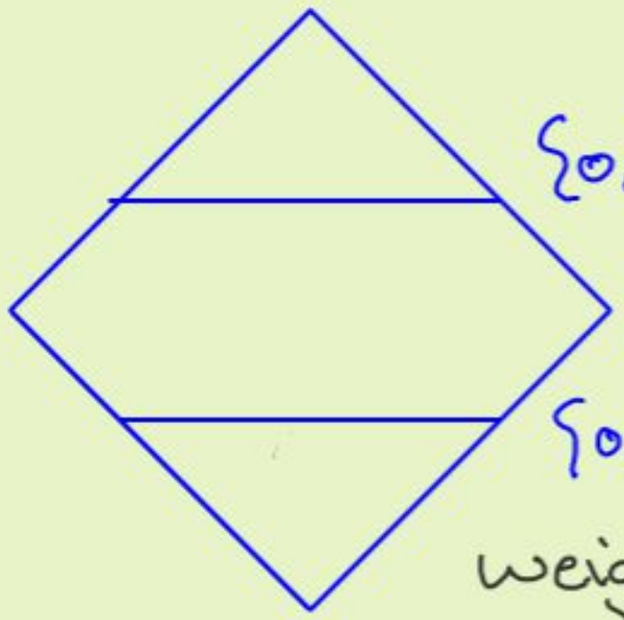

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$\mathbb{F} = \mathbb{Q}$: $R(x_1, \dots, x_n) = \left(\sum_i x_i - k\right)$

$\mathrm{char}(\mathbb{F}) = p$: Same construction if $p \nmid q$.

Hegedüs's lemma

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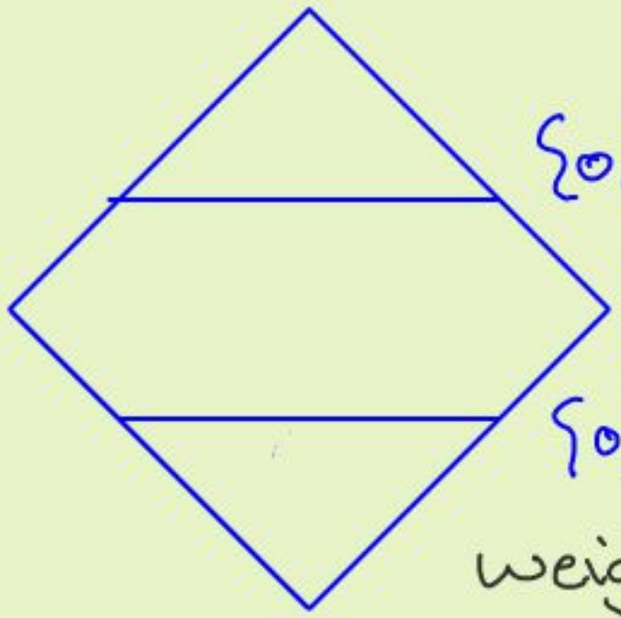

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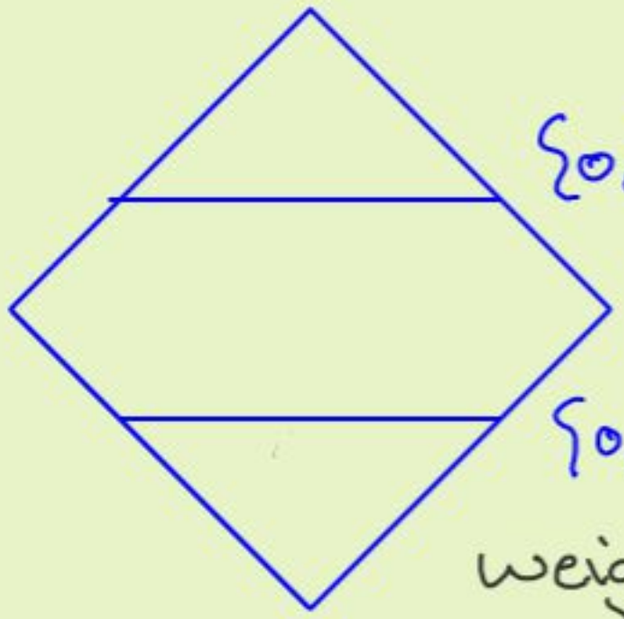

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Elementary symm. poly. of deg. q

$$R(x_1, \dots, x_n) = \sum_{|I| = q} x^I$$

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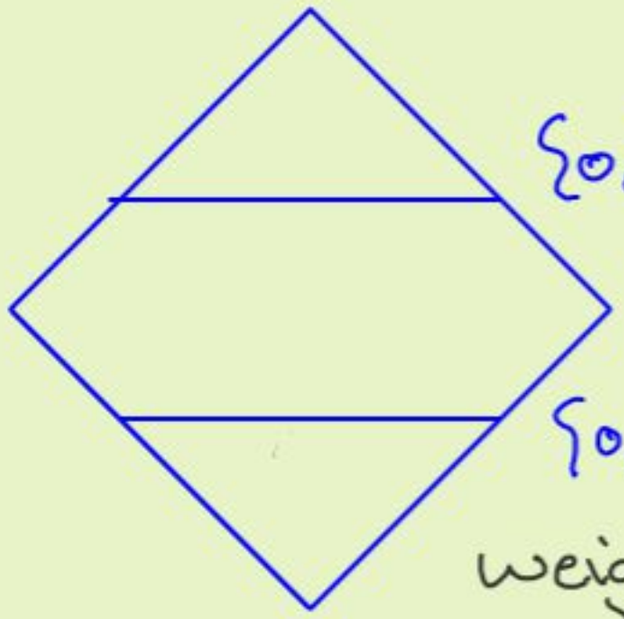

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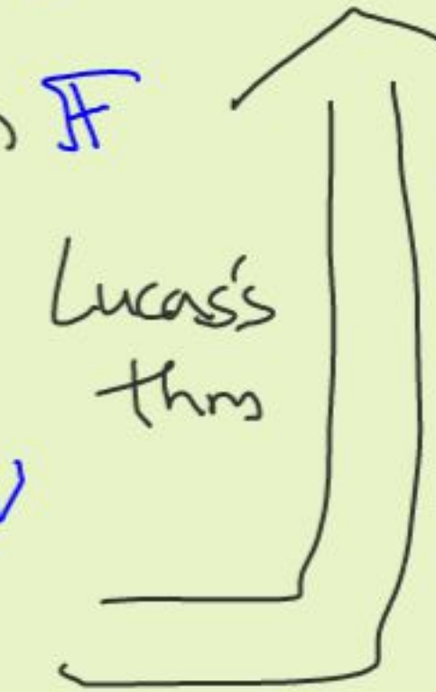

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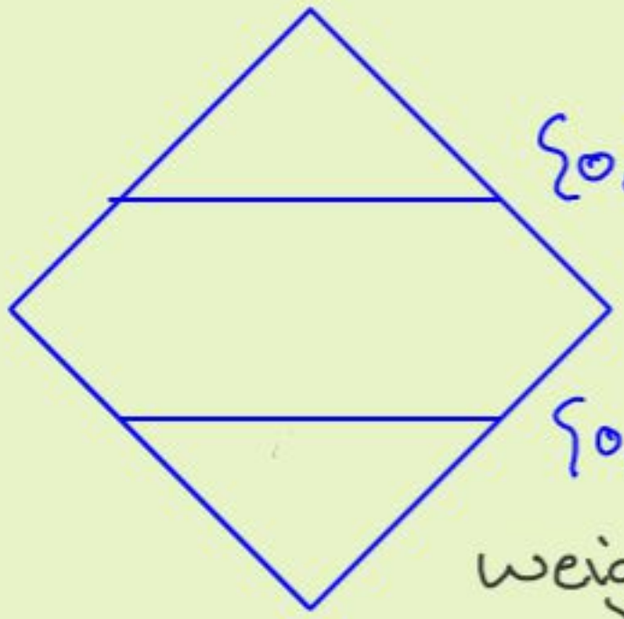

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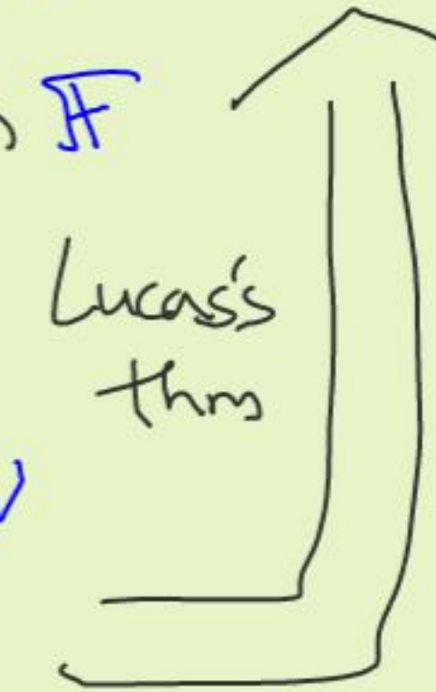

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Hegedüs's lemma: This is tight.

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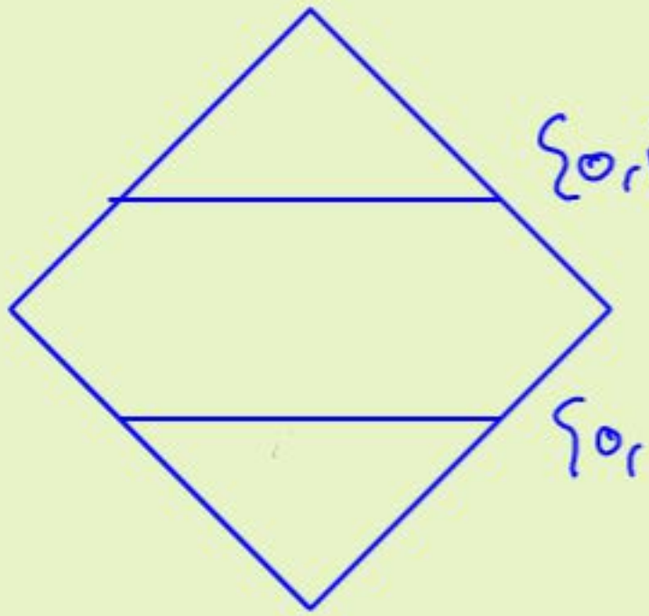

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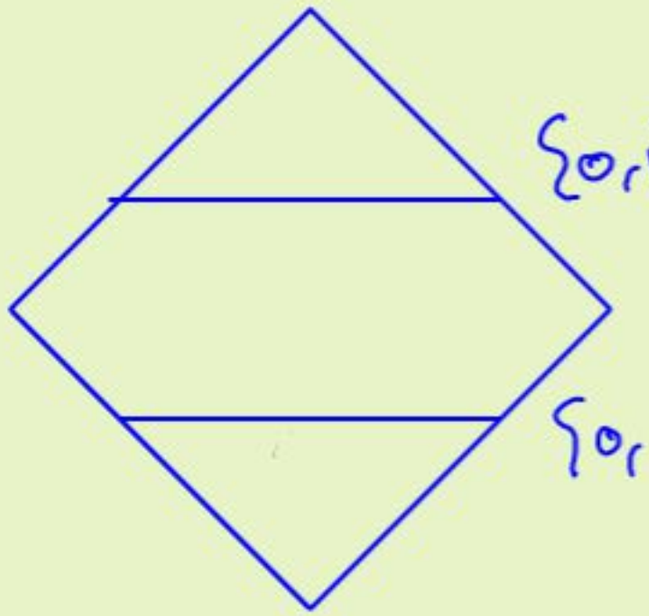

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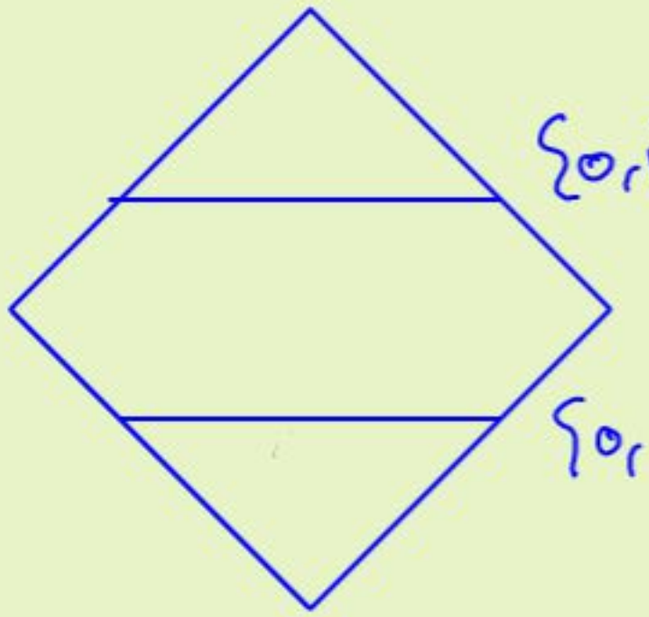

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[Hegedüs '10] - Gröbner bases

[S'18, Alon'19] - Elementary

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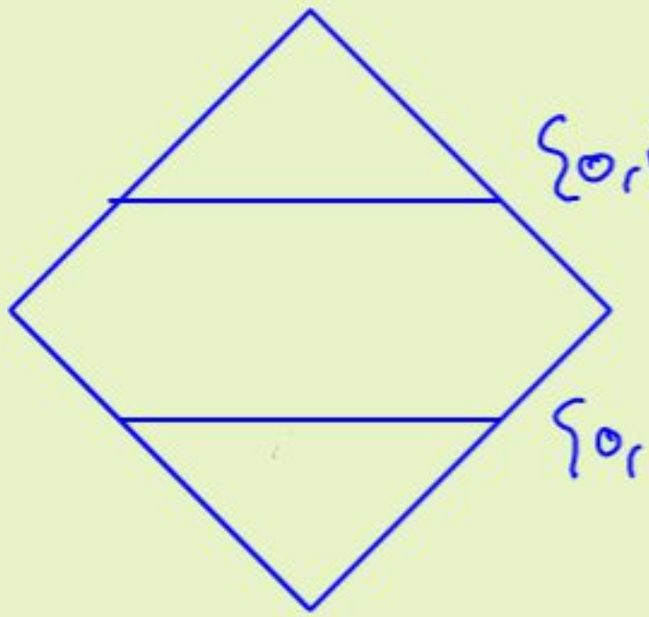

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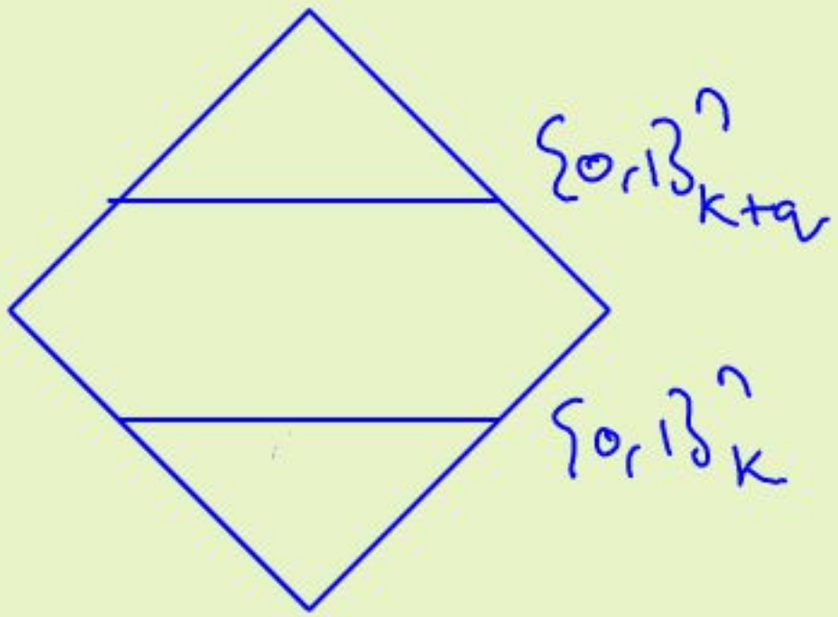
Applications

- Algebraic ckts [Alon-Kumar-Volk'18]
- Threshold ckts [Hrubeš et al. '19]

Robust Hegedüs



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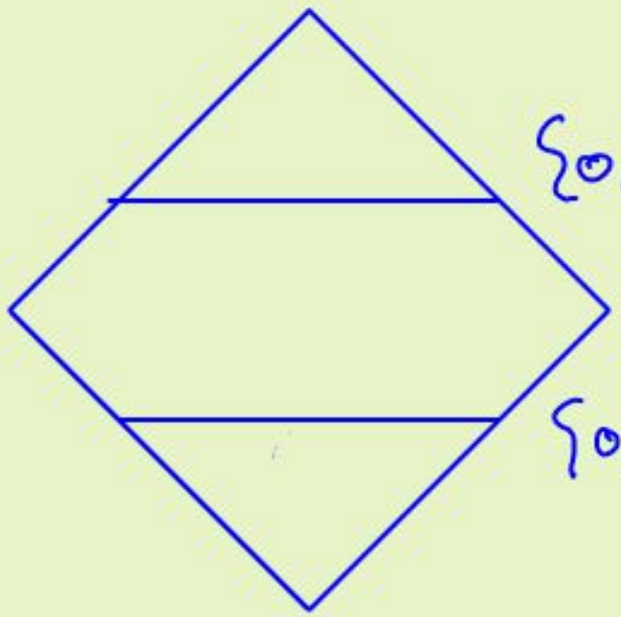

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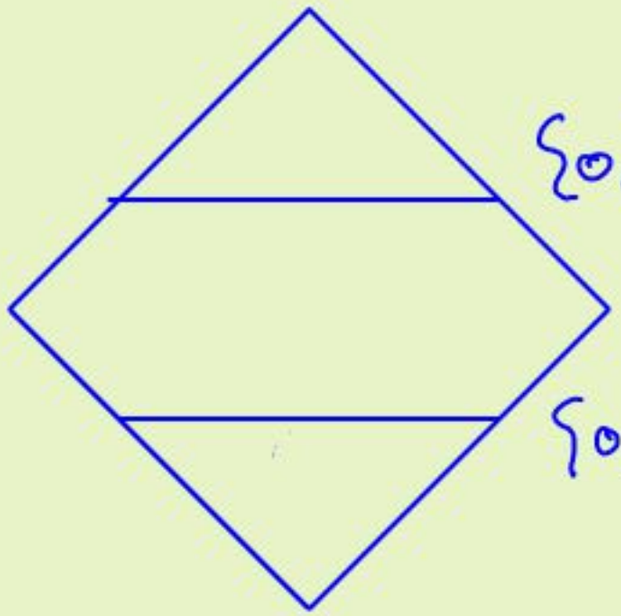

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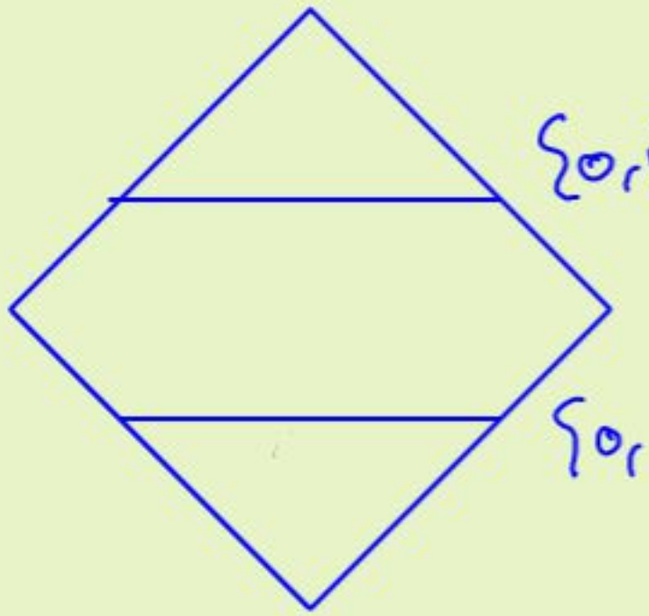

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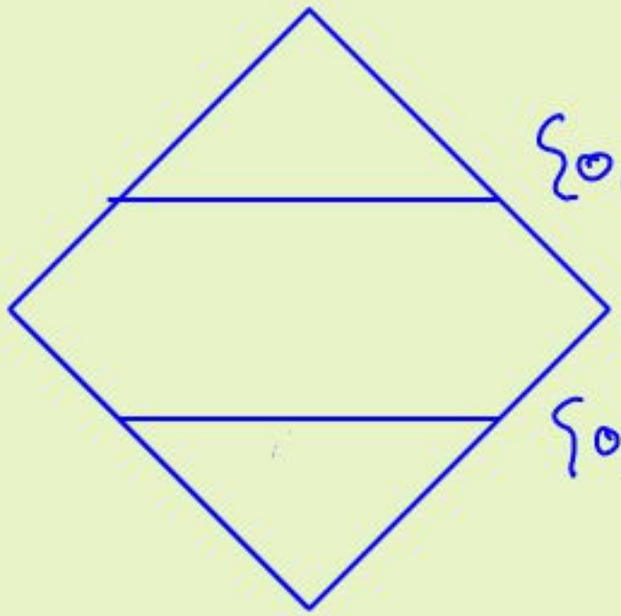
Same lower bound (i.e. q)?

No!

Sampling construction

$F[x_1, \dots, x_n]$



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Say $k = 0.49n$, $k+q = 0.51n$.

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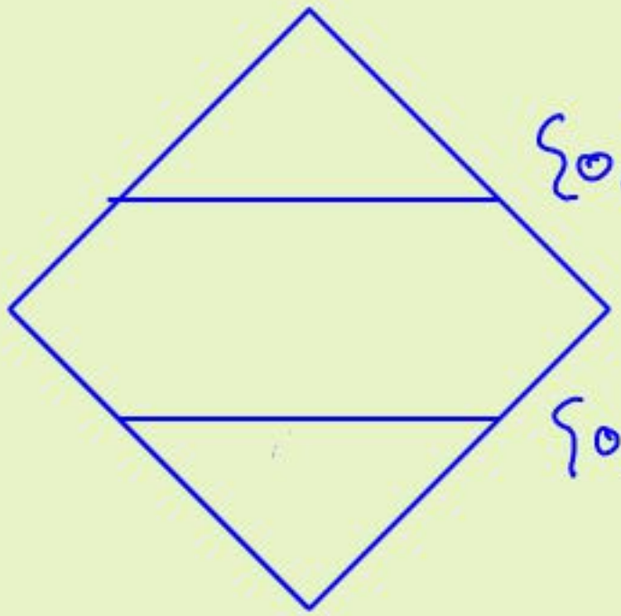

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$R(x_1, \dots, x_n) = \text{Maj}_t(x_1, \dots, x_t)$

Sampling construction

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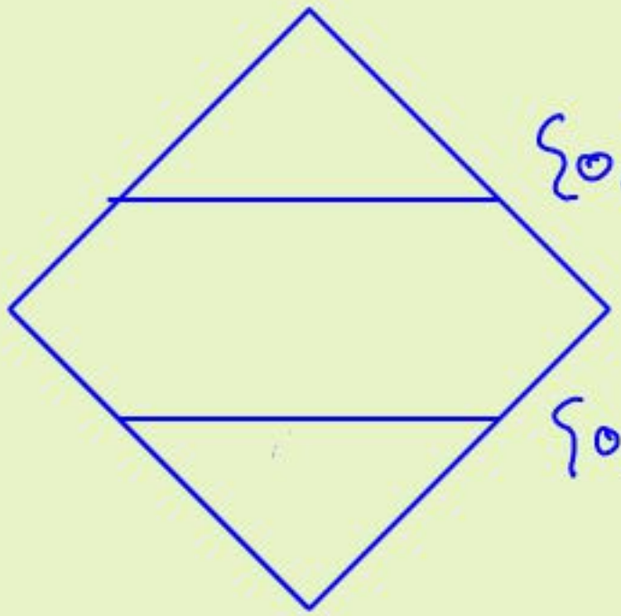

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(A) Satisfies above when $t = \Theta(\log 1/\varepsilon)$.

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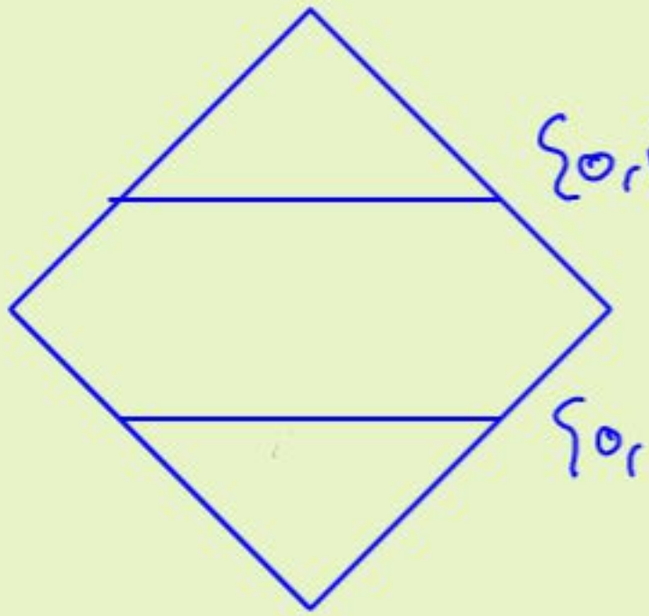

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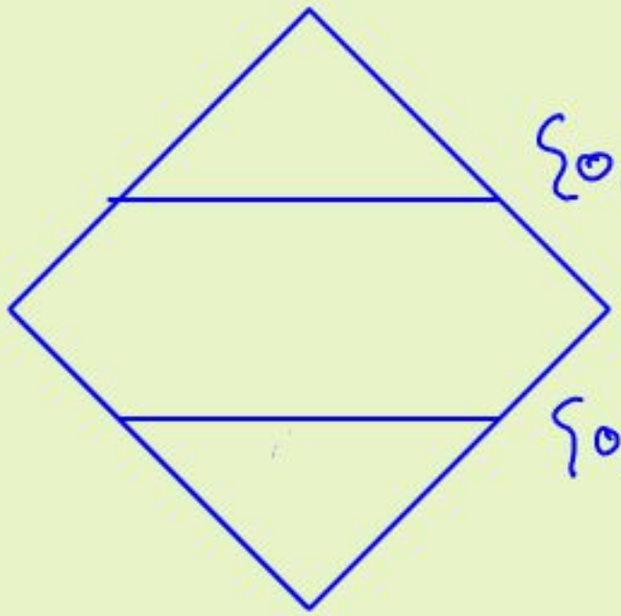
(B) deg $t = \Theta(\log 1/\varepsilon)$.

Main lemma

$\mathbb{F}[x_1, \cdots, x_n]$



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Main Result: $\deg(R) = \Omega(\min\{\text{sampling bound}, q\})$

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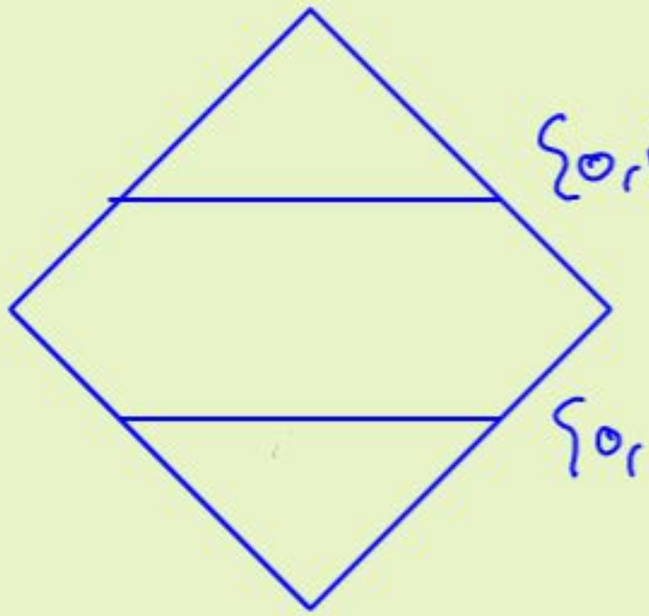

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Eg: $k = n/2 - \sqrt{n \log 1/\varepsilon}$

$k+q = n/2$

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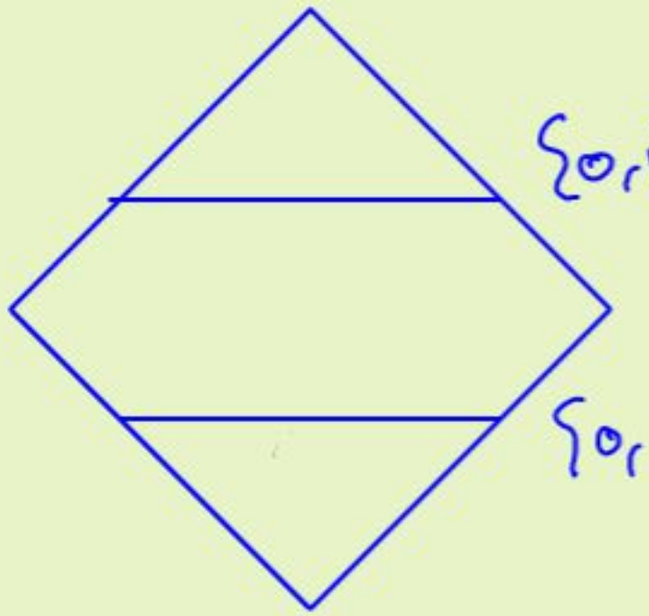

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(I) $\Pr_{|a|=k}[R(a) \neq 0] \leq \varepsilon$

(II) $\Pr_{|b|=k+q}[R(a) \neq 0] \geq 1-\varepsilon$

$\text{char}(\mathbb{F}) = p, \; q = p^s$

Main Result: $\deg(R) = \Omega(\min\{\text{sampling bound}, q\})$

Eg: $k = n/2 - \sqrt{n \log 1/\varepsilon} \Rightarrow \deg(R) = \Omega(\sqrt{n \log 1/\varepsilon})$

$k+q = n/2$

Applications

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→ Used to prove lower bounds for $AC^0[MOD_p]$ ckts.

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→ Algorithmic applications

Tight bounds on pdeg of any symmetric f?

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Theorem 8 (Known upper bounds on probabilistic degree of symmetric functions [STV19]). *Let $\mathbb{F}$ be a field of constant characteristic $p > 0$ and $n \in \mathbb{N}$ be a growing parameter. Let $f \in s\mathcal{B}_n$ be arbitrary and let (g, h) be a standard decomposition of f. Then we have the following for any $\varepsilon > 0$.*

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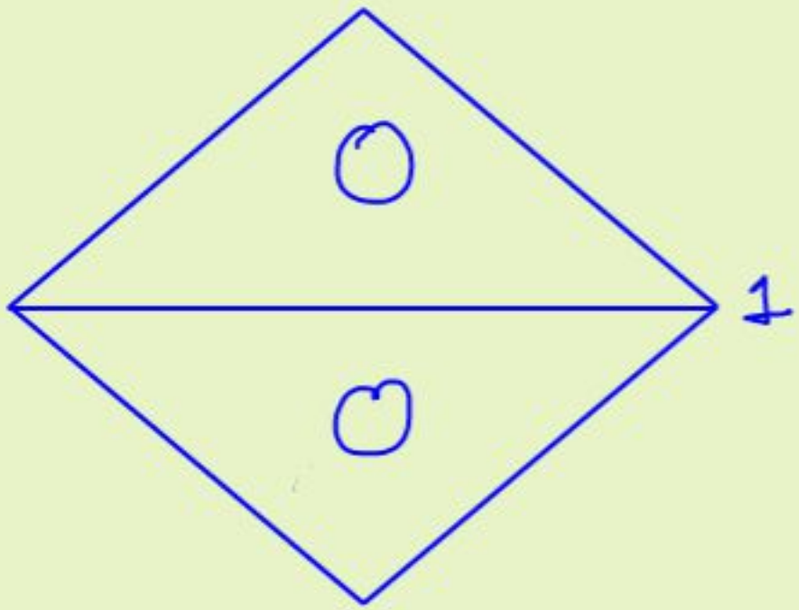
This paper: Tight bounds for all symmetric fns

Example: $ETHR_n^{n/2}$



$pdeg_\varepsilon(ETHR_n^{n/2}) = ?$

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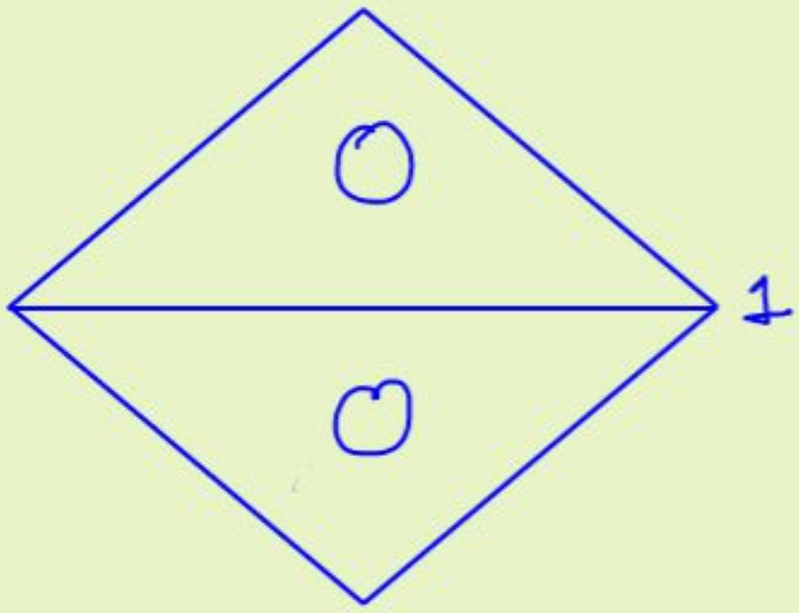

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[STV'19]:

$$Maj_{n/2}(x) = \sum_{i=0}^{n/4-1} ETHR_n^{n/2}(x 1^i 0^{n/2-i})$$

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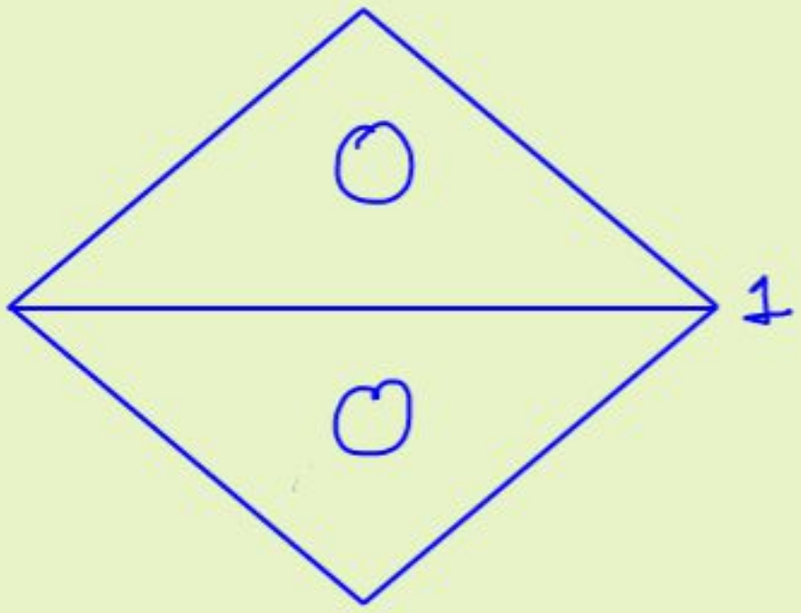

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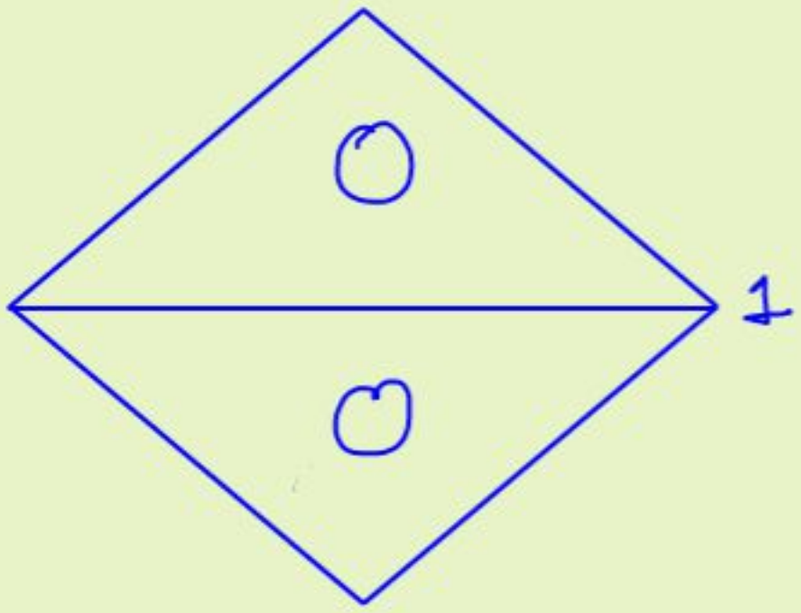

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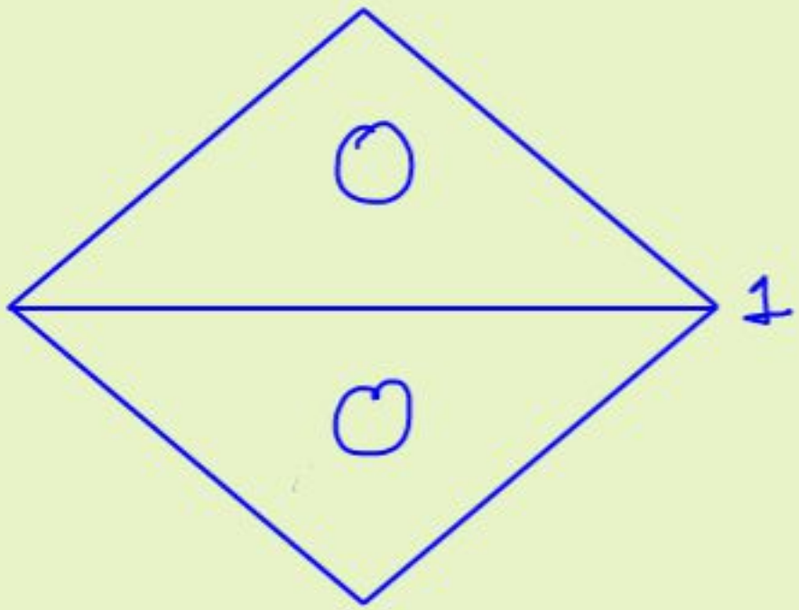

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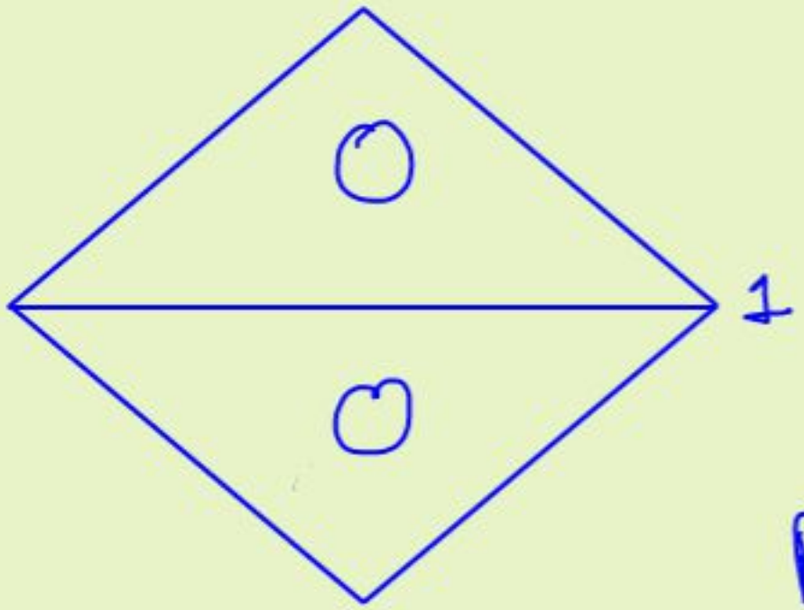

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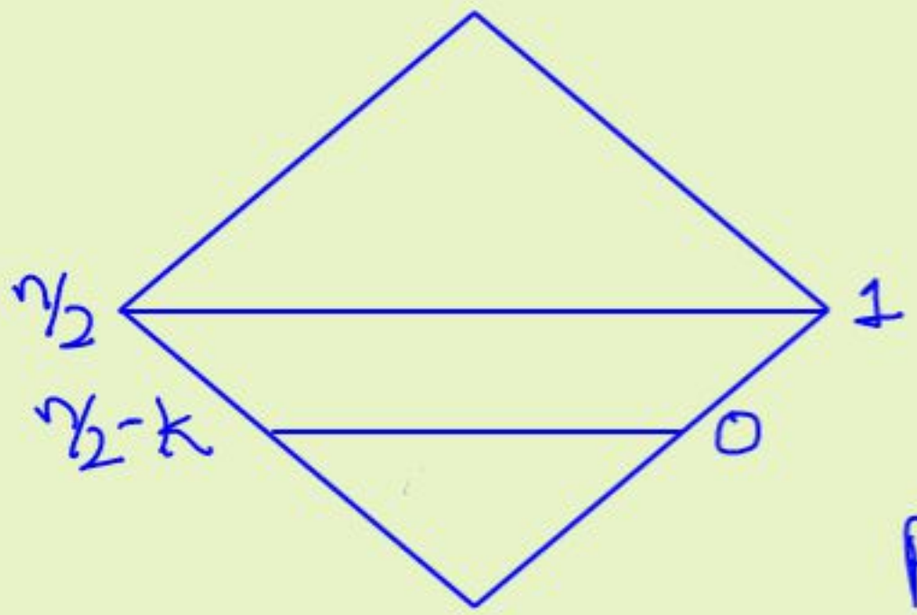

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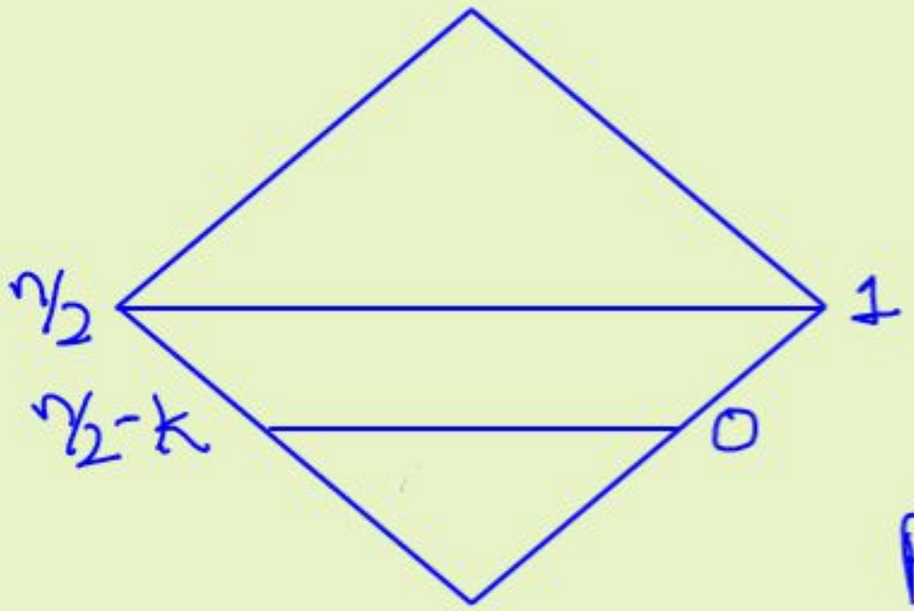

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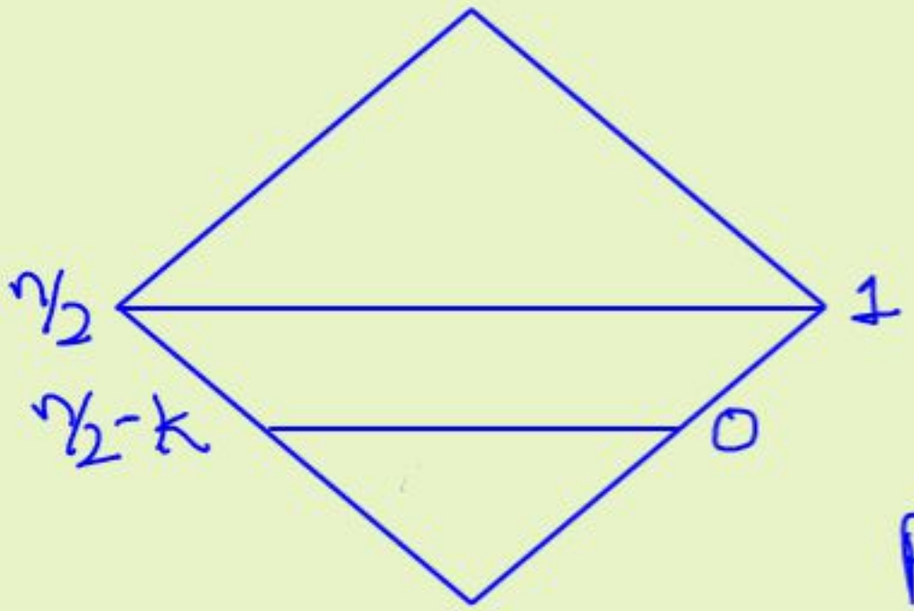

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$k := p^3 = \Theta(\sqrt{n})$ ⟹ $deg(P) = \Omega(\sqrt{n})$

Robust Hegedüs

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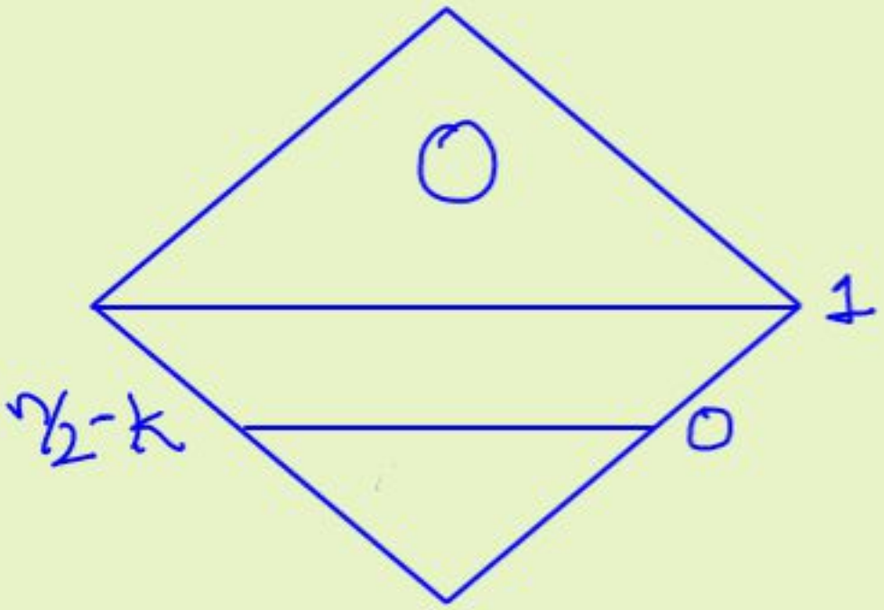

Message:

→ Robust Megedüs ⇒ lbds on pdeg of simple promise symmetric fns.

→ Easier to reduce to other symmetric functions.

Proof of Robust Hegedüs lemma

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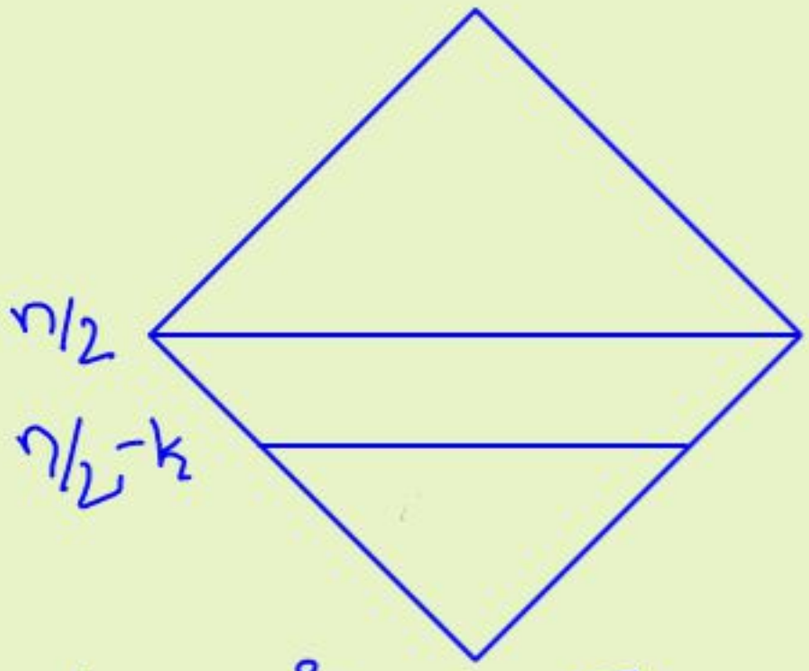


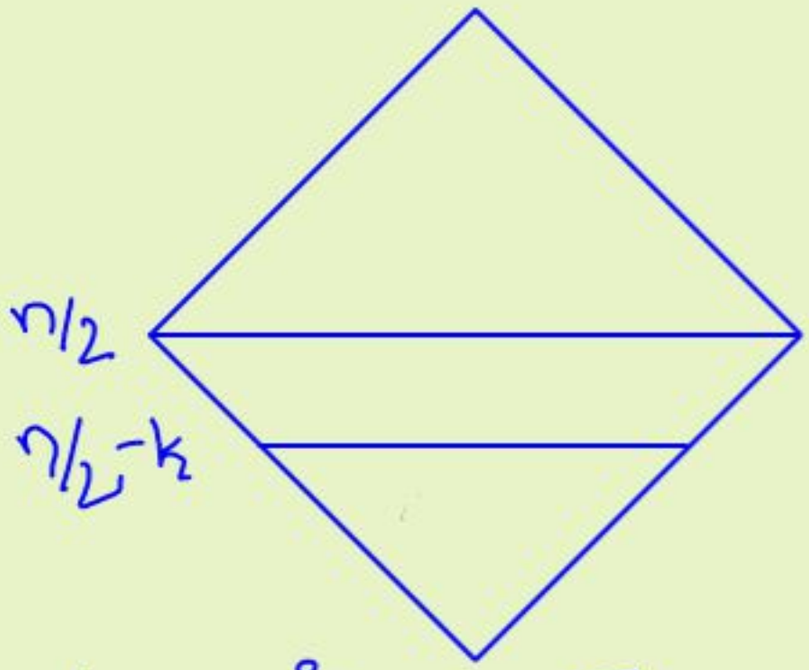
$R(x_1,\dots,x_n) \in \mathbb{F}_p[x_1,\dots,x_n]$ s.t.

(A) $\Pr_{a \sim \{0,1\}^n_{n/2}}[R(a) = 0] \leq \varepsilon$

(B) $\Pr_{b \sim \{0,1\}^n_{n/2}}[R(b) \neq 0] \leq \varepsilon \Rightarrow \deg(R) = \Omega(\sqrt{n})$

$k = p^3 = C \cdot \sqrt{n}$

Proof of Robust Hegedüs lemma



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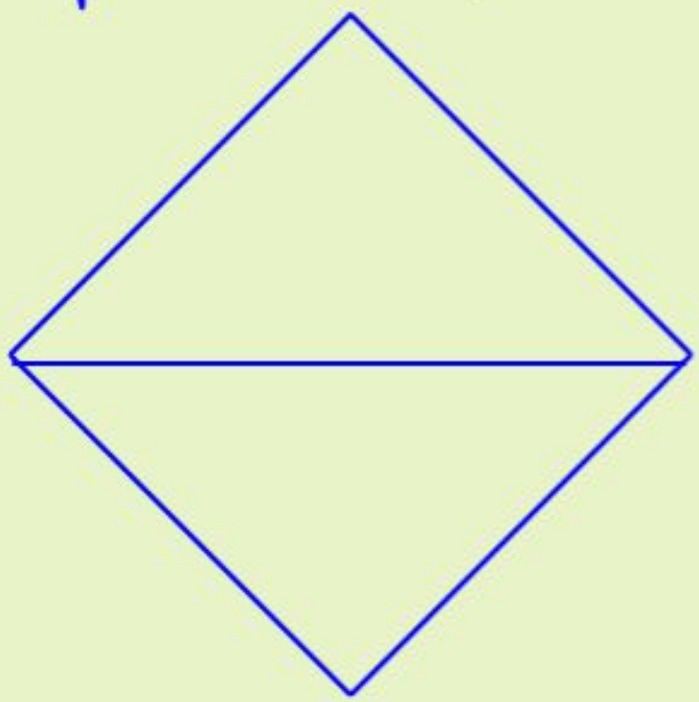
$K=p^3=C\cdot\sqrt{n}$



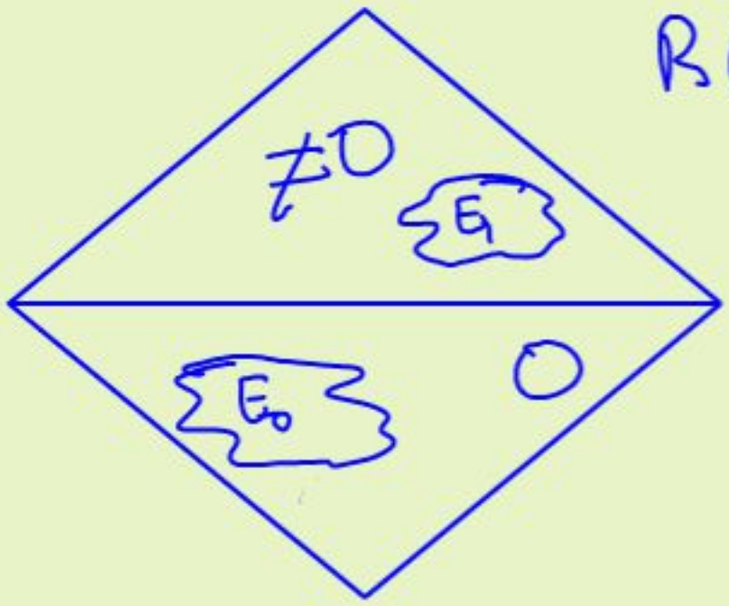
Smolensky's thm:

(A) $\Pr_{a\sim\{0,1\}^n_{\le n/2}}[R(a)=0] \le \varepsilon$

(B) $\Pr_{b\sim\{0,1\}^n_{<n/2}}[R(a)\neq 0] \le \varepsilon$

$\Rightarrow \deg(R)=\Omega(\sqrt{n})$.

Proof of Smolensky's thm

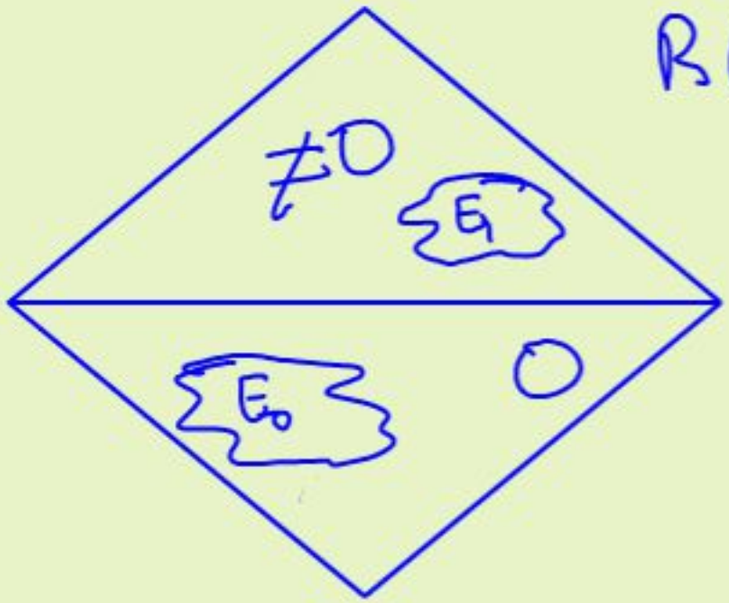


$R(x_1, \dots, x_n)$

$|E_0|, |E_1| \leq \varepsilon \cdot 2^n.$

$\Rightarrow \deg(R) = \Omega(\sqrt{n})$

Proof of Smolensky's thm



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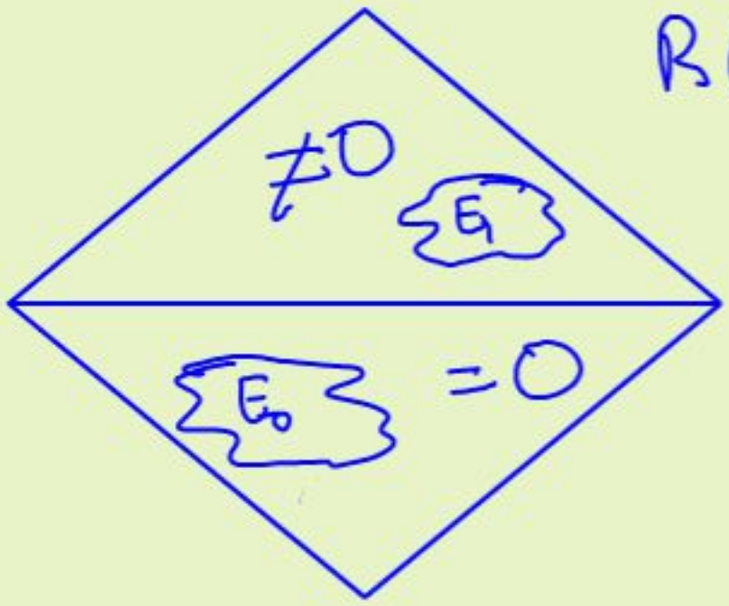
$\Rightarrow \deg(R) = \Omega(\sqrt{n})$

Clm 1: $\exists Q$, $\deg(Q) \leq n/2 - \Omega(\sqrt{n})$ s.t.

$Q|_{E_0 \cup E_1} \equiv 0$ & $Q \cdot R \not\equiv 0$

Proof of Smolensky's thm



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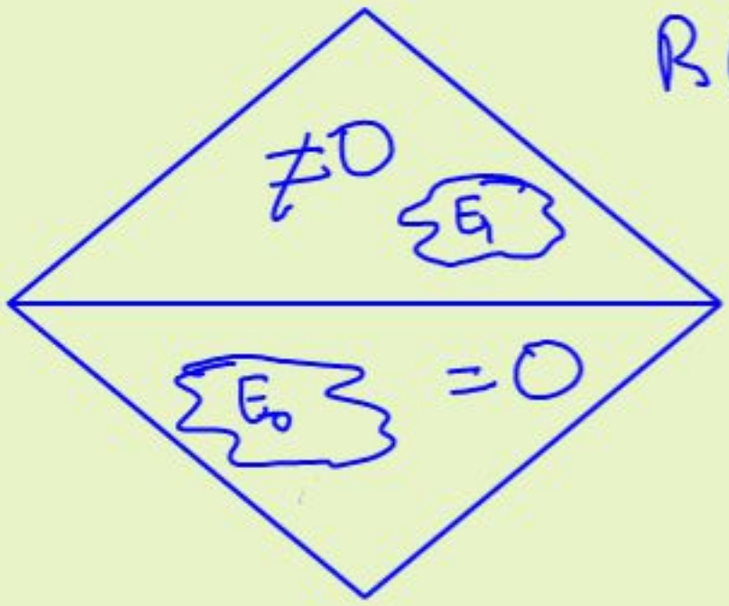
Clm 2: $P \not\equiv 0$ & $P|_{\Sigma^0_{\leq n/2} \cup \Sigma^1_{\leq n/2}} \equiv 0 \Rightarrow \deg(P) \geq n/2$.

Proof of Smolensky's thm



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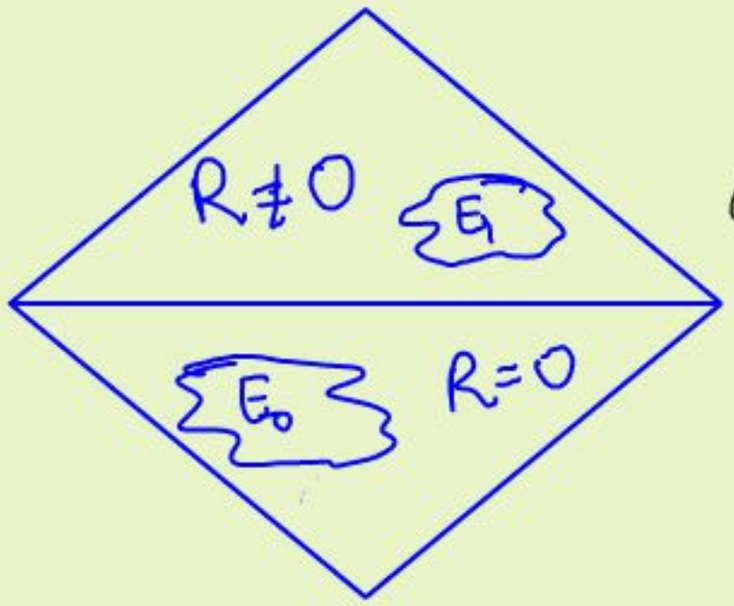
Clm 2: $P \not\equiv 0$ & $P|_{\Sigma^n_{0,1} \leq n/2} \equiv 0 \Rightarrow \deg(P) \geq n/2$.

Pf of Smolensky: $P = Q \cdot R \Rightarrow \deg(P) \geq n/2$. $\square$

Proof of Smolensky's thm

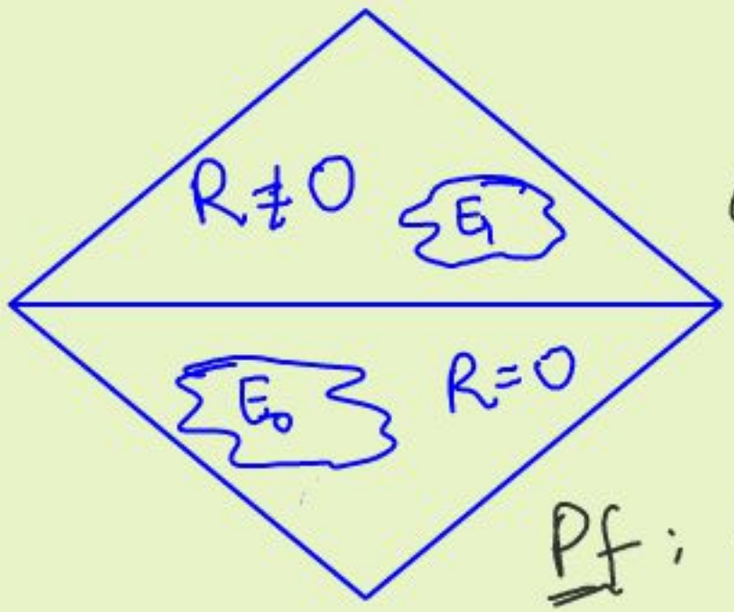


$E = E_0 \cup E_1$, $|E| \le 2\varepsilon \cdot 2^n$.

Clm 1: $\exists Q$, $\deg(Q) \le n/2 - \Omega(\sqrt{n})$

s.t $Q|_E \equiv 0$ & $Q \cdot R \not\equiv 0$.

Proof of Smolensky's thm



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Clm 1: $\exists Q$, $\deg(Q) \le n/2 - \Omega(\sqrt{n})$ s.t $Q|_E \equiv 0$ & $Q \cdot R \not\equiv 0$.

Pf: Assume Q of deg $\le D$.

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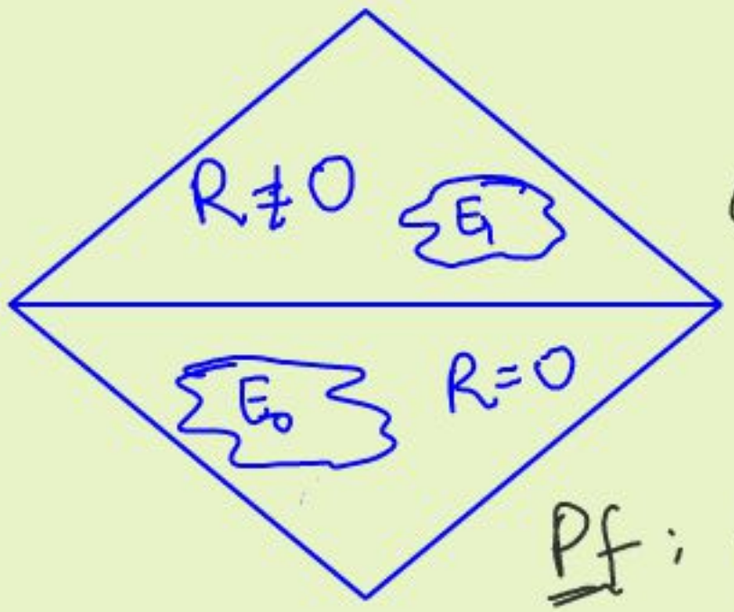

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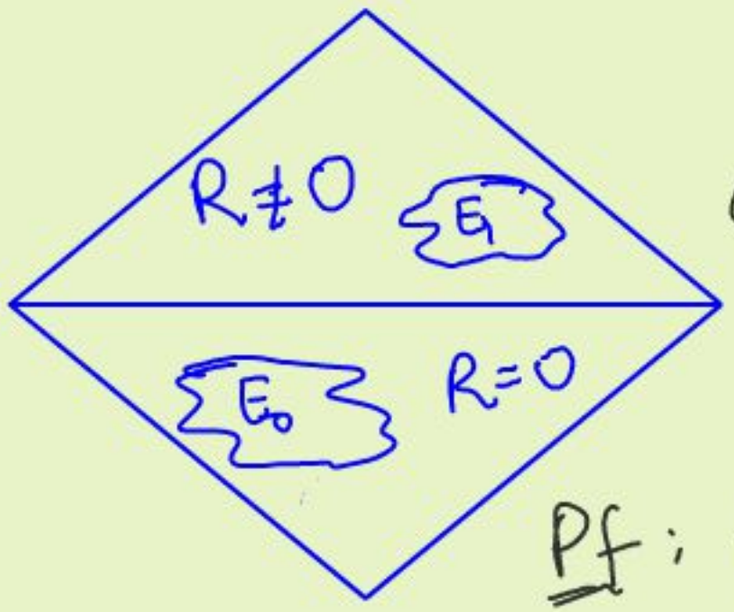

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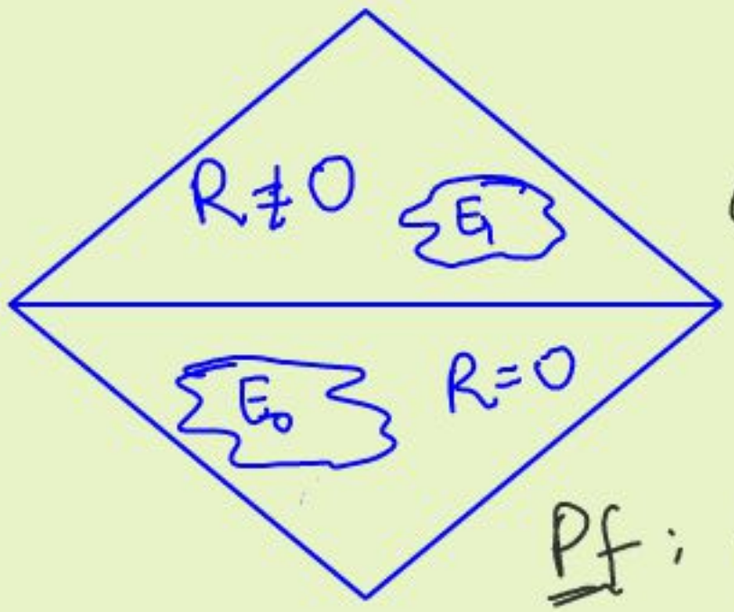

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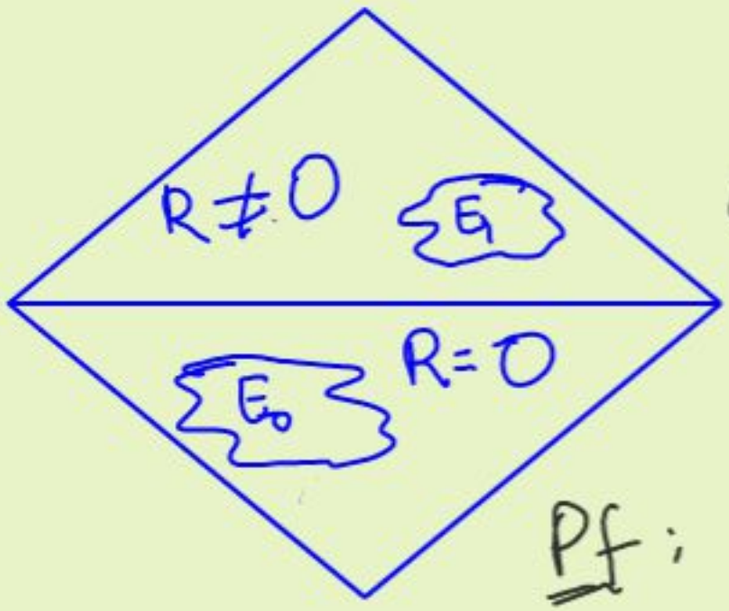

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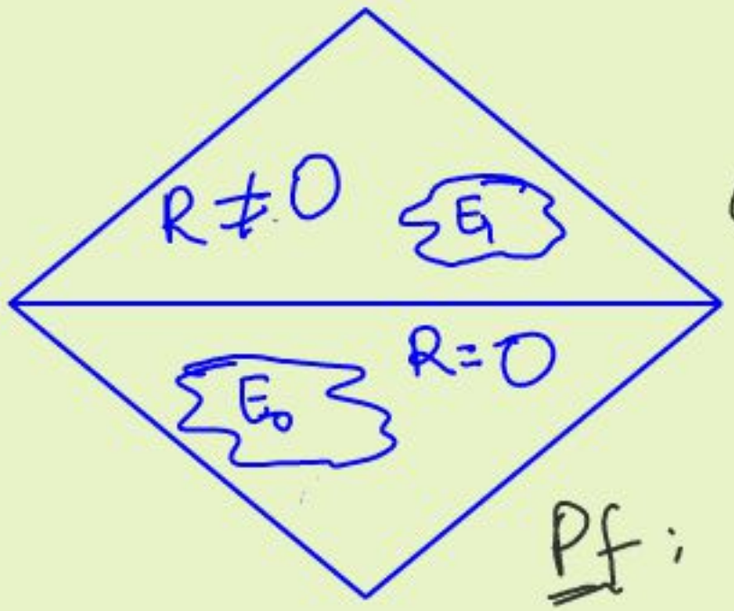

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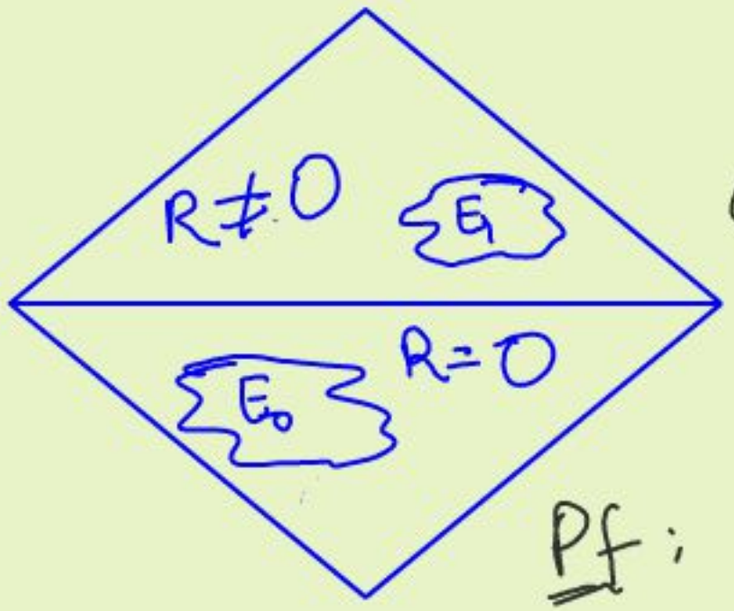

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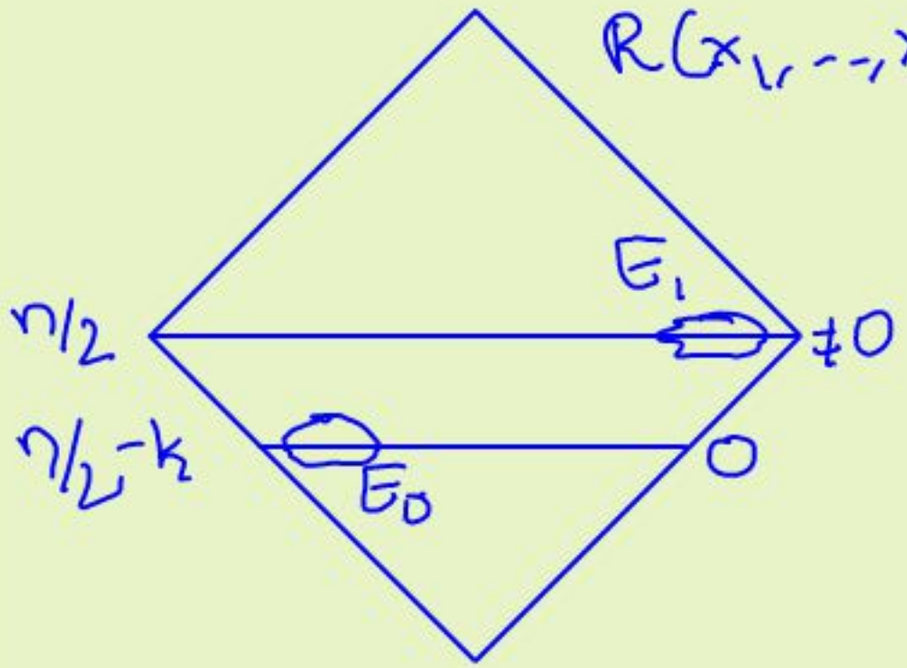
Proof of Robust Hegedüs lemma



$$k=\rho^3 \geq C\cdot\sqrt{n},\ C\to\infty$$

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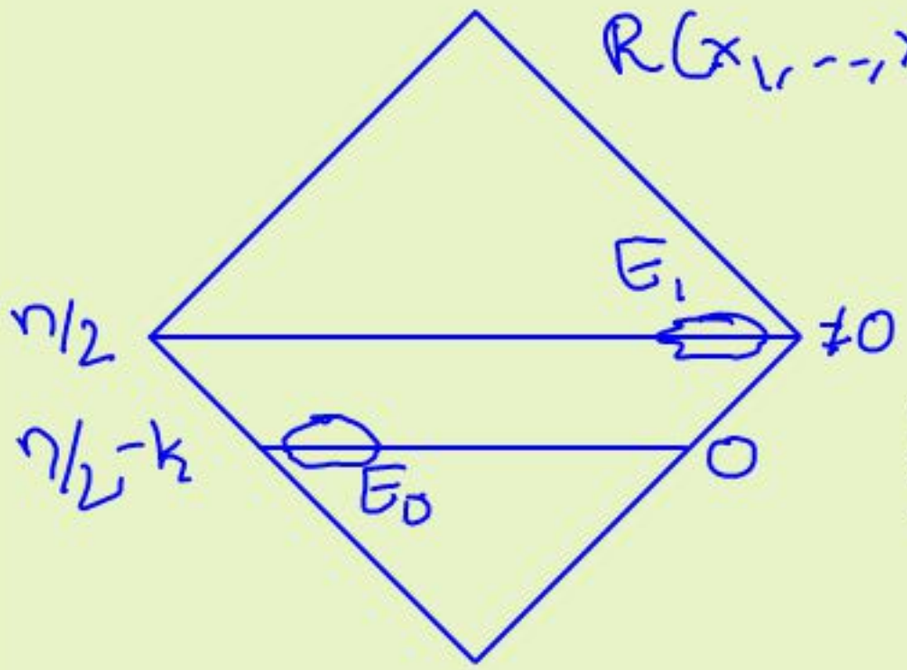

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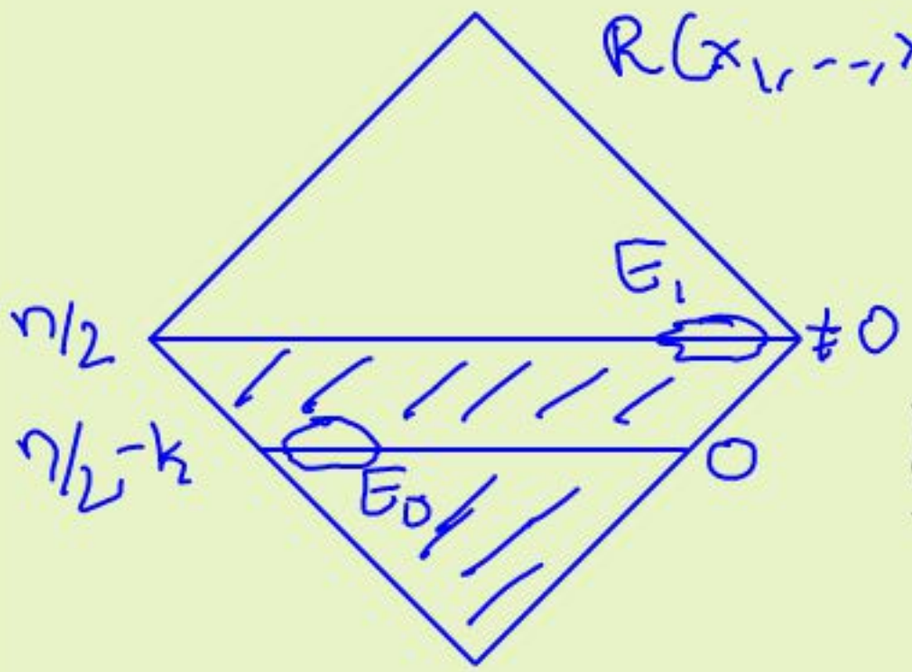

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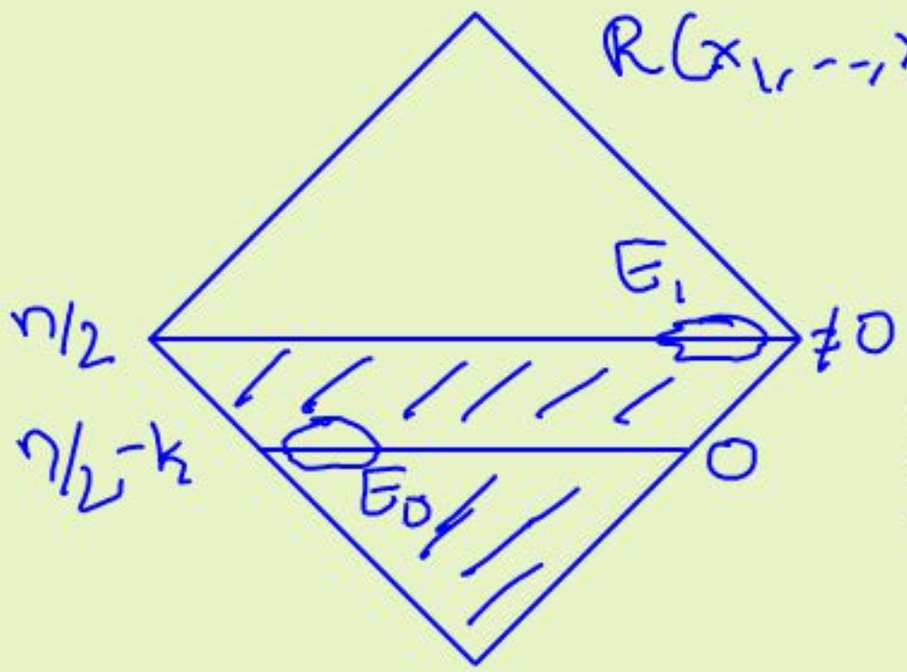

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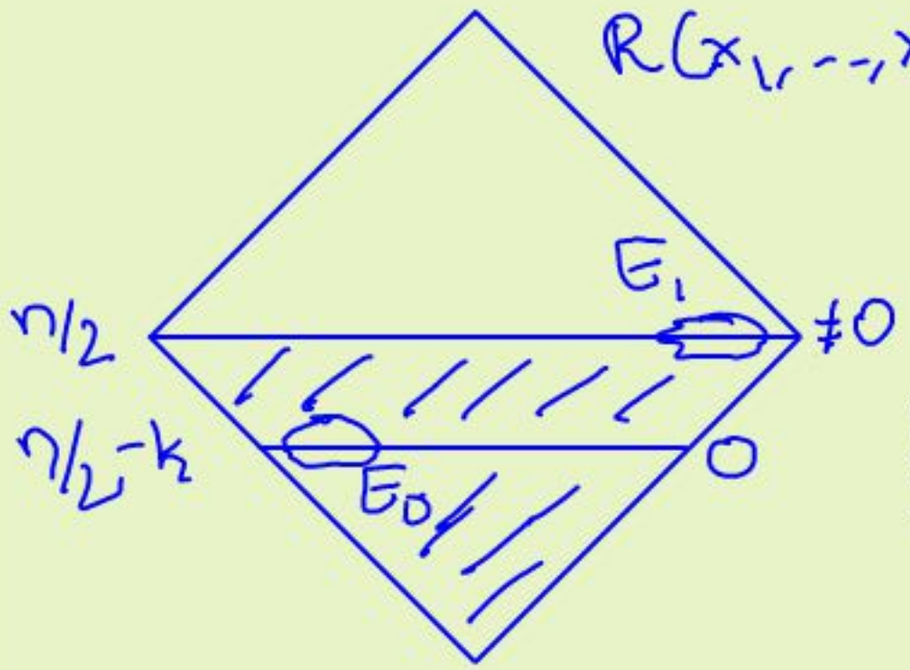

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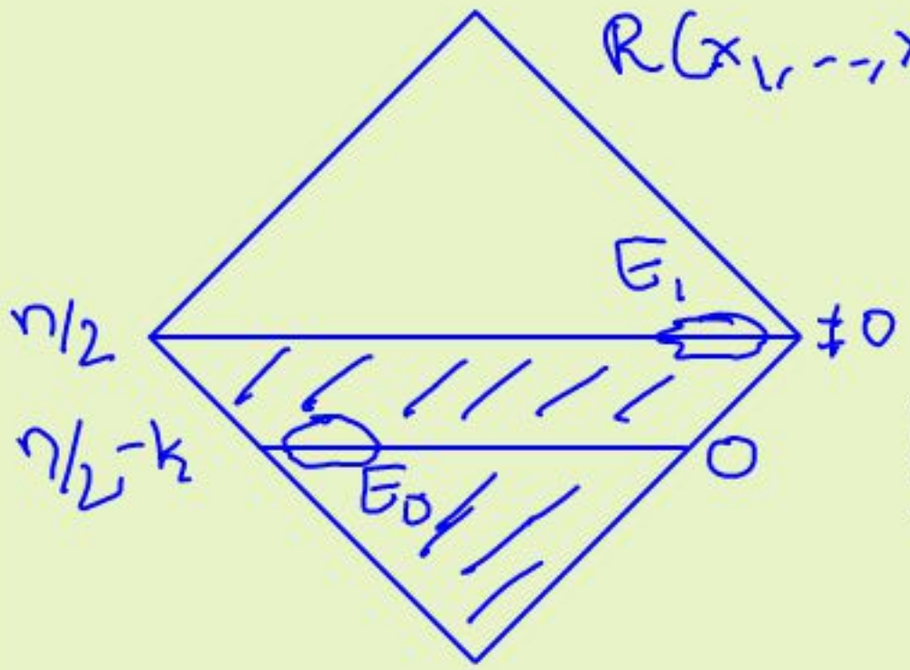

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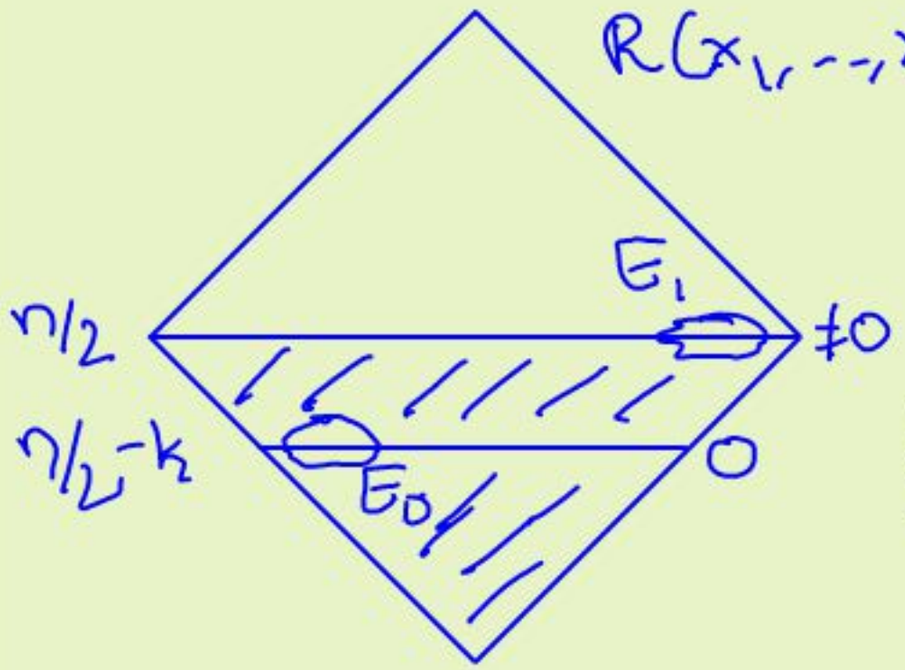

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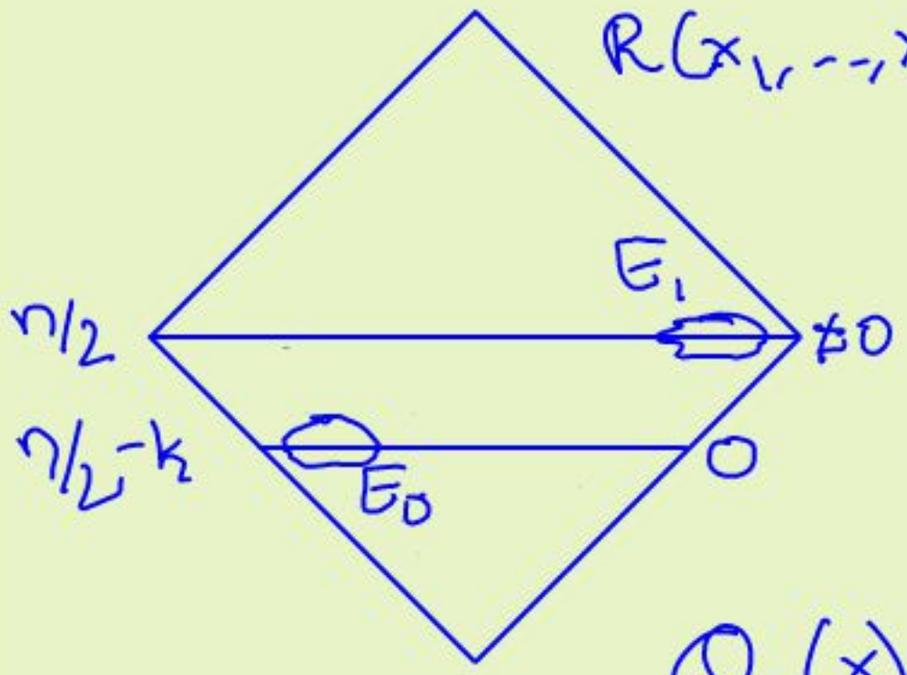

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$$Q_1(x) = \prod_{j=1}^{s-1}\left(\left(\sum_{|I|=p} x^I\right)^{p-1} - 1\right)$$

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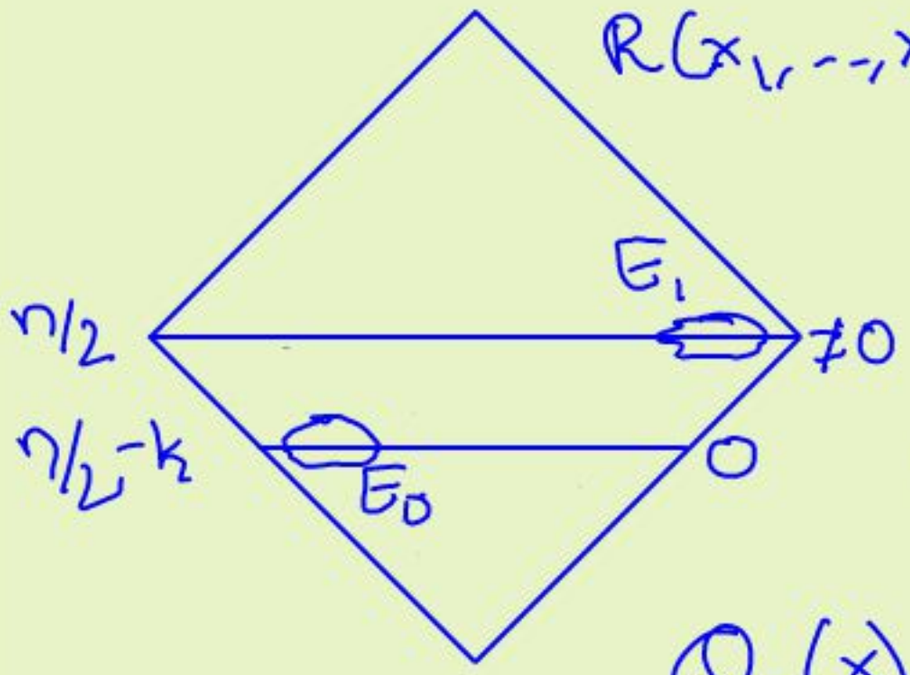

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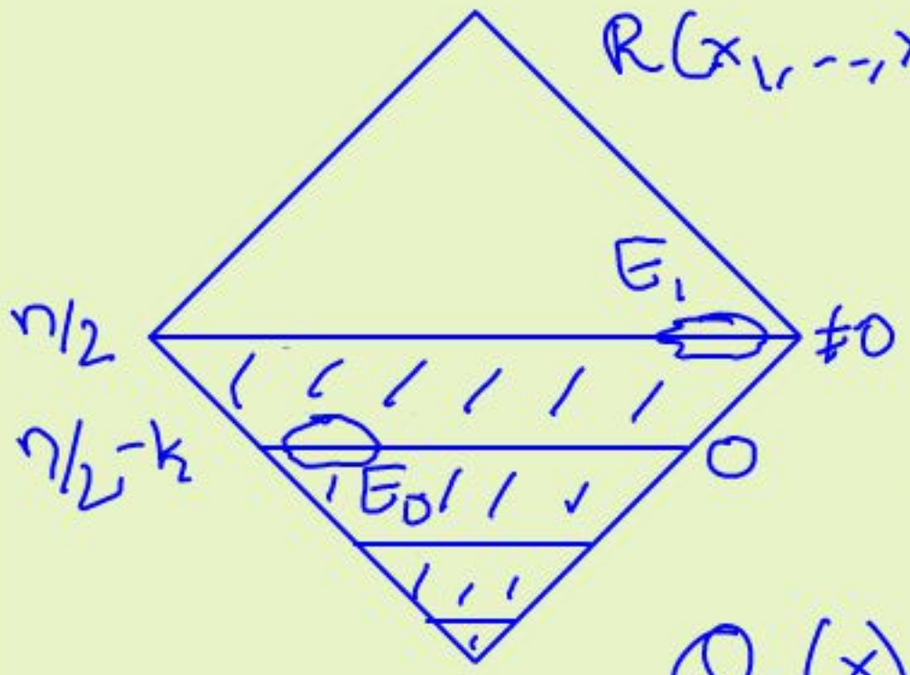

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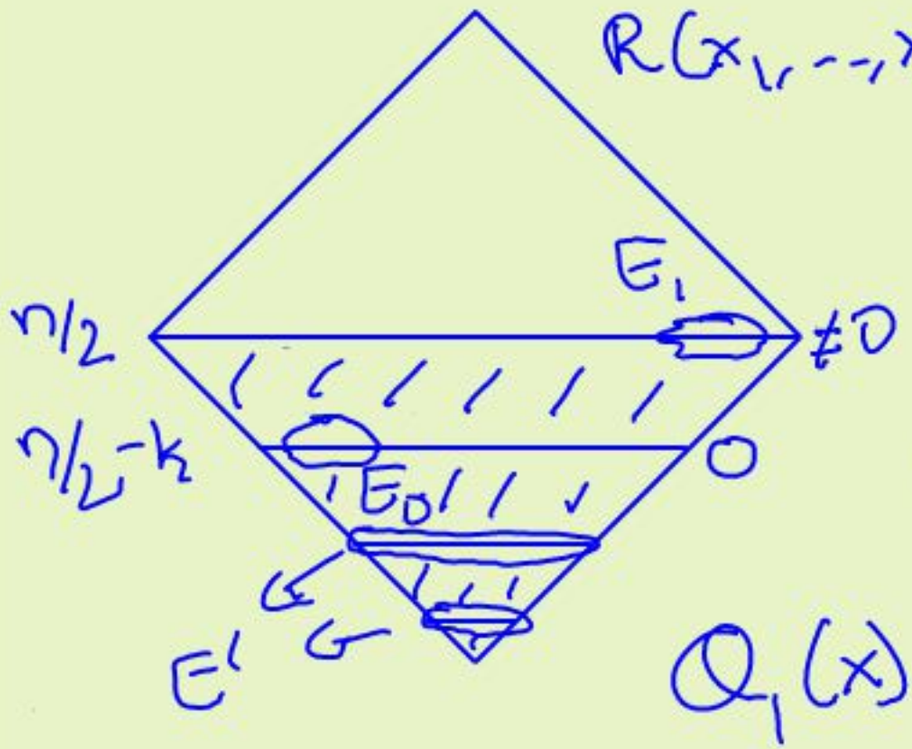

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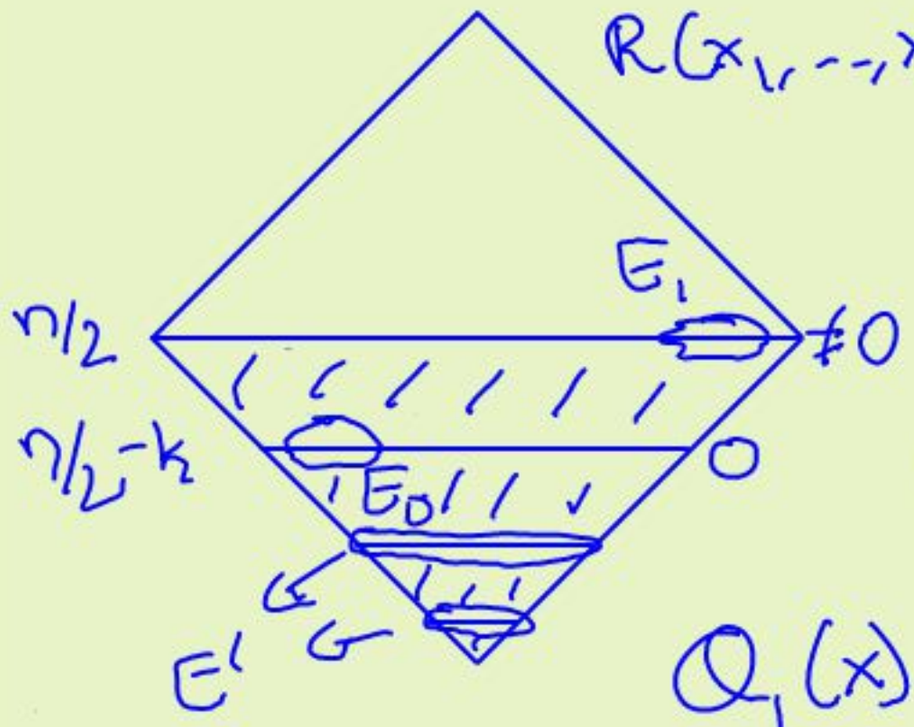

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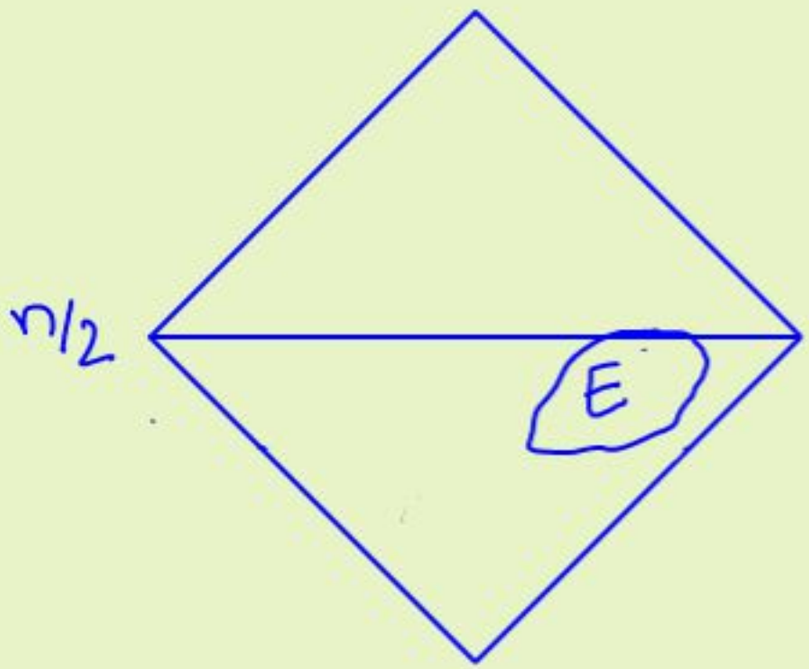
$E'' = E \cup E' \Rightarrow |E''| \le \varepsilon \cdot \binom{n}{n/2} + \binom{n}{n/2-2k} + \binom{n}{n/2-3k} + \cdots$

$\le 2\varepsilon \cdot \binom{n}{n/2}$

Proof of Robust Hegedüs lemma

$E \subseteq \{0,1\}^n, \ |E| \leq \varepsilon \cdot \binom{n}{n/2}$



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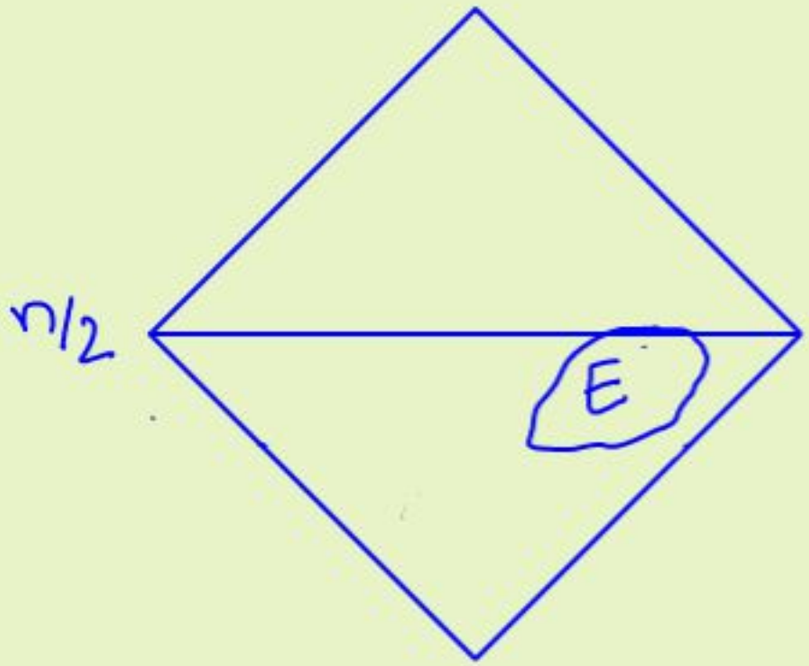

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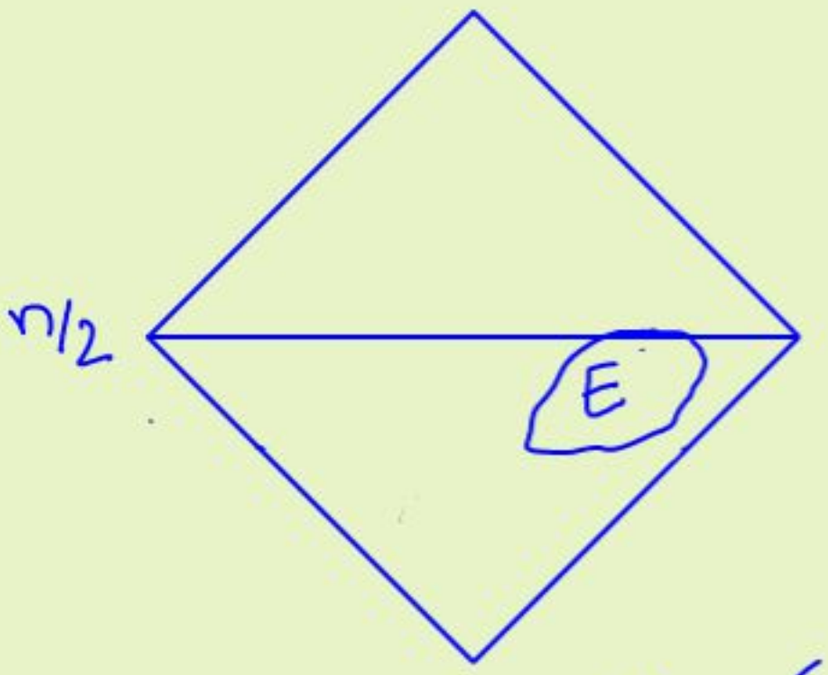

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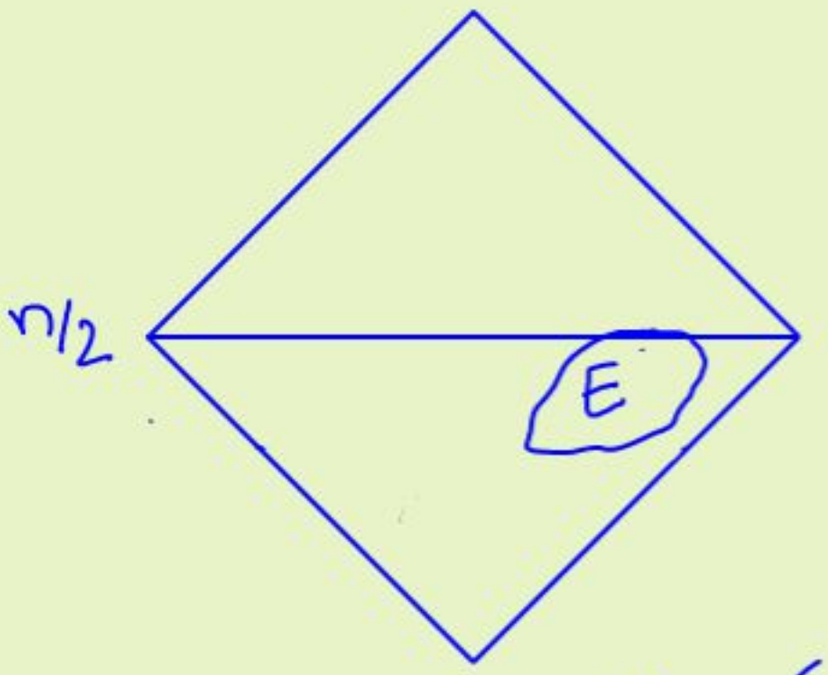

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Thm: [Nie-Wang '14]

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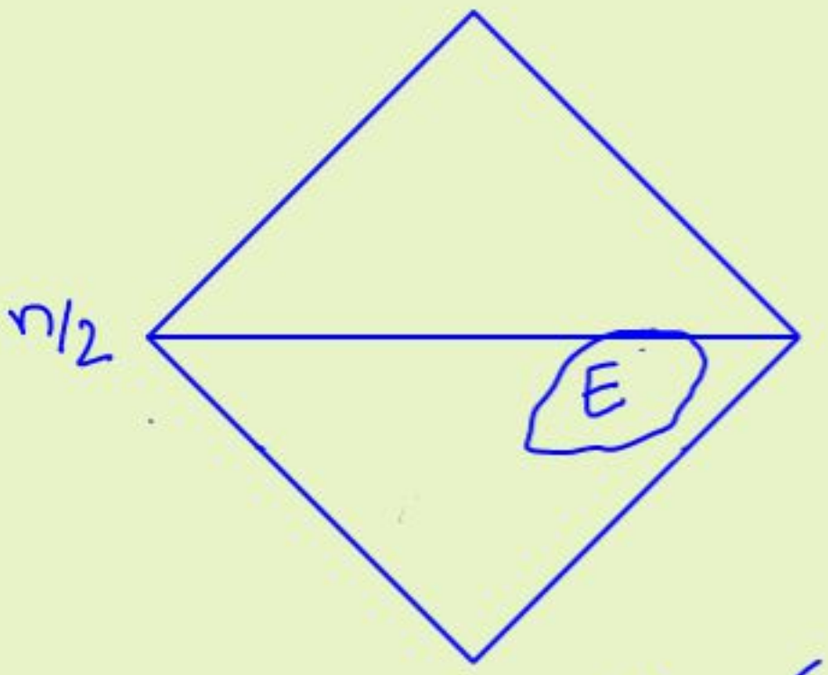

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→ # of constraints

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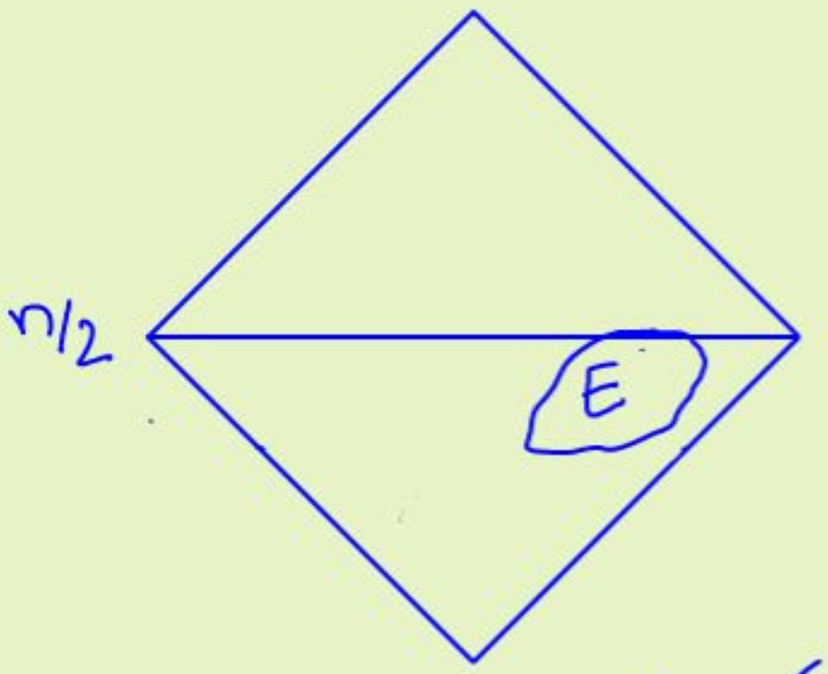

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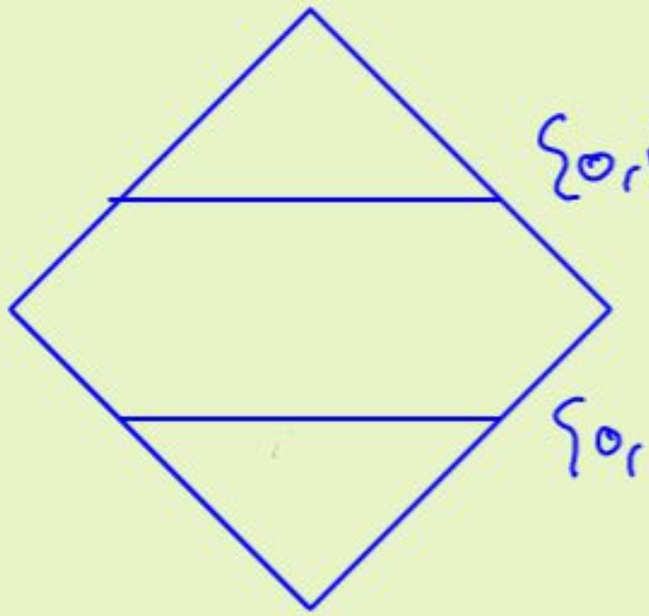
$$\frac{|Cl_D(E)|}{2^n} \leq \frac{|E|}{\binom{n}{\leq D}} \leq \frac{\varepsilon \cdot \binom{n}{n/2}}{\Omega(2^n)} \leq O(\varepsilon) \cdot \binom{n}{n/2}$$

□

Main lemma

$\mathbb{F}[x_1, \cdots, x_n]$



Q: lowest deg. $R(x_1, \cdots, x_n)$ s.t.

(I) $\Pr_{|a|=k}[R(a) \neq 0] \leq \varepsilon$

(II) $\Pr_{|b|=k+q}[R(a) \neq 0] \geq 1-\varepsilon$

$\mathrm{char}(\mathbb{F}) = p, \; q = p^s$

Main Result: $\deg(R) = \Omega(\min\{\text{sampling bound}, q\})$

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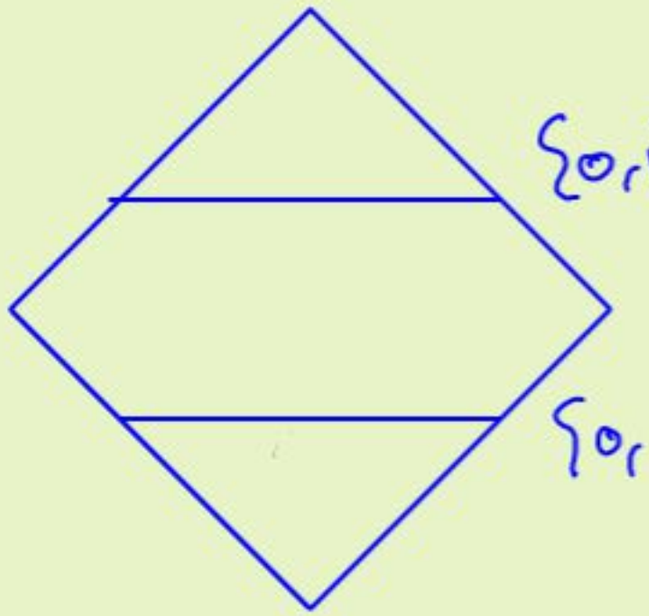

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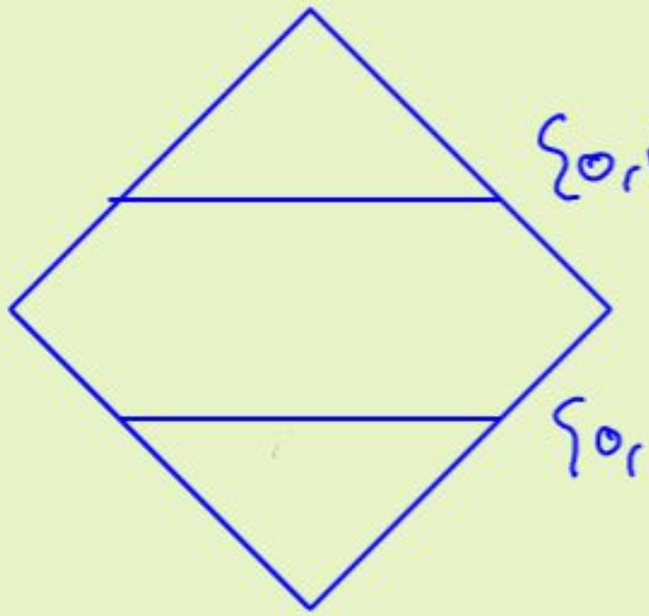
Applications: Degree lower bds for polynomial approximations.

Main lemma



$\mathbb{F}[x_1, \cdots, x_n]$

Q: lowest deg. $R(x_1, \cdots, x_n)$ s.t.

(I) $\Pr_{|a|=k}[R(a) \neq 0] \leq \varepsilon$

(II) $\Pr_{|b|=k+q}[R(a) \neq 0] \geq 1-\varepsilon$

$\mathrm{char}(\mathbb{F}) = p$, $q = p^s$

Main Result: $\deg(R) = \Omega(\min\{\text{sampling bound}, q\})$

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Others?

Thanks!

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→ Read-once Branching Programs - [Brody-Verbin '10]

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Minimum degree of R?

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$\varepsilon = \Omega(1) \Rightarrow \deg(R) = \Omega(\frac{1}{\delta})$

[Limaye et al. '19] :

[Chattopadhyay et al. '19]

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[Chattopadhyay et al. '19] Smaller ε?

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Corollary to Robust Hegedüs: $\deg(R) = \Omega\left(\frac{1}{\delta}\log\frac{1}{\varepsilon}\right)$.