Majority is incompressible by $\mathsf{AC}^0[p]$ circuits

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Part 1
Background, Examples, and Motivation

Basic Definitions

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$\mathrm{Majority} = \{\mathrm{Majority}_n\}_{n \in \mathbb{N}}$, where $\mathrm{Majority}_n \colon \{0,1\}^n \to \{0,1\}$.

$\mathrm{Majority}_n(x_1, \ldots, x_n) = 1$ if and only if $\sum_i x_i \geq n/2$.

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No explicit lower bounds for poly size circuits beyond depth $\log n / \log\log n$.

Technique does not generalize to modulo m gates, where $m = p \cdot q$.

As far as we know, it is possible that $\mathrm{NP} \subseteq \mathrm{AC}^0_3[6]$ (linear size).

This Talk

Understand structure of polynomial-size circuits with mod p gates computing **Majority.**

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Follows from the investigation of more general framework: "Interactive Compression Games".

Hybridizes computational complexity and communication complexity.

Example: Boolean circuits for symmetric functions

Idea. Boolean circuits can process $\log n$ bits very efficiently.
Every $f\colon \{0,1\}^{\log n} \to \{0,1\}$ computed by CNF/DNF of size n.

Circuit for $\text{Majority}_n(x)$. Computes $O(\log n)$-bit string counting #1's in x.

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Partition input bits into $(\log n)$-bit blocks, produce $(\log \log n)$-bit strings from each block.

In each layer, reduces number of strings by a factor of roughly $\log n$.

Example: Boolean circuits for symmetric functions

Lemma. For every $d \geq 1$, we obtain an AC_d^0 circuit with $n/(\log n)^{(d-1)-o(1)}$ output wires encoding #1's in x.

n input bits processed in $O(\log_{\log n} n) = O(\log n / \log\log n)$ stages.

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We will revisit this construction later in the talk.

Interactive Compression Games (Chattopadhyay and Santhanam, 2012)

Fix a circuit class $\mathcal{C}$ and a Boolean function f.
We define a communication game between Alice and Bob.

Alice knows the input $x \in \{0,1\}^n$, but her computations are limited to $\mathcal{C}$.

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Minimize total number of bits sent by Alice.

$f \notin \mathcal{C} \iff$ $\mathcal{C}$**-compression game for** f **is nontrivial.**

Interactive Compression Games

Formally:

A $\mathcal{C}$-bounded protocol $\Pi_n = \langle C^{(1)}, \ldots, C^{(r)}, f^{(1)}, \ldots, f^{(r-1)}, E_n \rangle$ with $r = r(n)$ rounds consists of a sequence of $\mathcal{C}$-circuits for Alice, a strategy for Bob, given by functions $f^{(1)}, \ldots, f^{(r-1)}$, and a set of accepting transcripts E_n.

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Every protocol Π_n has its signature$(\Pi_n) = (n, s_1, t_1, s_2, \ldots, t_{r-1}, s_r)$, which is the sequence corresponding to the input size $n = |x|$ and the length of the messages exchanged by Alice and Bob during the protocol.

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Π_n solves the compression game of a function $h_n \colon \{0,1\}^n \to \{0,1\}$ if

$$h(x) = 1 \quad \Longleftrightarrow \quad \text{transcript}_{\Pi_n}(x) \in E_n.$$

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Finally, we let cost$(\Pi_n) = s_1 + \ldots + s_r$.

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Fortnow and Santhanam, 2008. conditional lower bound for instance compression.

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Chattopadhyay and Santhanam, 2012. Optimal lower bound for $\mathcal{C} = \mathrm{AC}^0$, $f =$ Parity. Partial results for $\mathrm{AC}^0[p]$-compression.

Applications and Motivation

Results have found applications in cryptography, parameterized complexity theory, PCPs, circuit lower bounds.

Our main motivation:

Understand information bottlenecks in circuit lower bounds.

Understand structure of optimal circuits/algorithms.

Interactive Compression versus Computation

$\text{InnerProduct}_n(x, y) \stackrel{\text{def}}{=} \sum_i x_i \cdot y_i \pmod 2$.

Threshold gate: $\sum_j w_i z_i \geq^? t, \quad w_j, t \in \mathbb{R}$.

Proposition [HMPSP'93]. InnerProduct $\notin$ poly(n)-TH $\circ$ poly(n)-TH.

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Proposition [HMPSP'93]. InnerProduct $\notin$ poly(n)-TH $\circ$ poly(n)-TH.

On the other hand,

Proposition. There exists a (poly(n)-TH $\circ$ poly(n)-TH)-compression game for InnerProduct with $O(\log n)$ rounds and communication cost $O(\log n)$.

Interactive Compression versus Computation

Protocol.
Alice's circuits are of the form $C(x, y, v)$.

(first layer) C computes $z_i \stackrel{\text{def}}{=} x_i \wedge y_i$, for every $i \in [n]$.

(second layer) C outputs $\text{sign}(\sum_{i\in[n]} z_i - \sum_{i\in[n]} v_i)$.

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Bob sends $v = 0^{n/2} 1^{n/2}$:
bit computed by Alice reveals if $\sum_{i \in [n]} x_i \cdot y_i$ is at least $n/2$.

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Bob sends string corresponding to the next step of the binary search, and so on.

Part 2: Main Results

Main Result

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Chattophadyay and Santhanam, 2012.
Any single-round $\mathsf{AC}^0_d[p]$-compression game for Majority requires communication $\sqrt{n}/(\log n)^{O(d)}$.

Main Result

[Theorem 1]. There exists a fixed constant $c \in \mathbb{N}$ such that, for each $d \in \mathbb{N}$, and every $n \in \mathbb{N}$ sufficiently large, the following holds.

1) Any $\mathsf{AC}^0_d[p]$-compression game for $\mathsf{Majority}_n$ (any number of rounds) has communication cost $\geq n/(\log n)^{2d+c}$.

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1) Any $\mathsf{AC}^0_d[p]$-compression game for Majority_n (any number of rounds) has communication cost $\geq n/(\log n)^{2d+c}$.

2) There exists a single-round $\mathsf{AC}^0_d[p]$-compression game for Majority_n with communication cost $\leq n/(\log n)^{d-c}$.

Lower bound against circuits with oracle gates

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Example:

[IW'97] $\exists f \in$ EXP that requires circuits of size $2^{\Omega(n)}$ then P $=$ BPP.

[KvM'99] $\exists f \in$ NE $\cap$ coNE that requires circuits with SAT-oracles of size $2^{\Omega(n)}$ then AM $=$ NP.

Lower bound against circuits with oracle gates

Lemma. Let C be a Boolean circuit over n variables from $\mathcal{C}_d(\text{poly}(n))$ augmented with oracle gates $f_i \colon \{0,1\}^{s_i} \to \{0,1\}^{t_i}$, where $i \in [r]$, for some $r = r(n)$.

Let $s = s_1 + \ldots + s_r$ be the total fan-in of these oracle gates, and $h \colon \{0,1\}^n \to \{0,1\}$ be the Boolean function computed by C.

Then h admits a $\mathcal{C}_d(\text{poly}(n))$-compression game with communication cost $c(n) \leq s$ consisting of at most $r + 1$ rounds.

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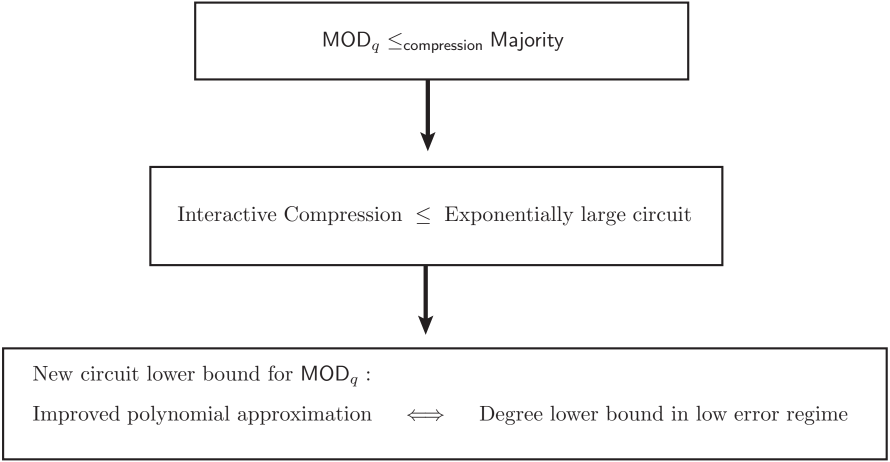
Main lower bound holds for protocols with unlimited number of rounds:

Corollary. If Majority is computed by an $\mathrm{AC}^0_d[p]$ circuit with arbitrary oracle gates, then the total fan-in of the oracle gates is $\geq n/(\log n)^{2d+O(1)}$.

Sketch of the lower bound (Theorem 1)

Let $\mathcal{C} = \mathsf{AC}^0_d[p]$, and consider a fixed prime $q \neq p$.



Compressing symmetric functions using Majority

Lemma.
Let $h\colon \{0,1\}^n \to \{0,1\}$ be an arbitrary symmetric function, $\mathcal{C}$ be a circuit class, and $d \geq 1$.

Assume that the $\mathcal{C}_d(\text{poly}(n))$-compression game for Majority_n can be solved with cost $c(n)$ in $r(n)$ rounds.

Then the $\mathcal{C}_{d+O(1)}(\text{poly}(n))$-compression game for h can be solved with cost $c_h(n) = O(c(2n) \cdot \log n)$ in $r_h(n) = O(r(2n) \cdot \log n)$ rounds.

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Proof sketch.
1) Compression for Majority implies compression for Th_k.
2) Alice and Bob perform a binary search.

From interactive compression to very large circuits

Proposition.

If there exists a $\mathcal{C}_d(\text{poly}(n))$-compression game for f_n with cost $c(n)$, then there exist circuits $C_1, \ldots, C_T$ from $\mathcal{C}_{d+O(1)}(\text{poly}(n))$, where

$$T \leq 2^{c(n)},$$

such that $\forall x \in \{0,1\}^n$,

$$f_n(x) = \bigvee_{i \in [T]} C_i(x).$$

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Depth blow-up is minimal: "Parallel simulation of all rounds".

The difficulty of analyzing very large circuits

Goal.
Lower bound against circuits of depth $d + O(1)$ and size $\geq 2^{c(n)}$.
Want to set $c(n) \approx n/\text{poly}(\log n)$.

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Idea.

$$f_n(x) = \dot{\bigvee_{i \in [T]}} C_i(x).$$

Initial function is a disjoint union of (poly-size) circuits C_i.

If $f(x) = 1$ then exactly one circuit evaluates to 1.

From interactive compression to very large circuits

Proposition (updated)

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such that $\forall x \in \{0,1\}^n$,

$$f_n(x) = \dot{\bigvee_{i \in [T]}} C_i(x) \quad \textbf{(“uniqueness property”)}$$

New circuit lower bound for MOD_q

Proposition.
For every $d \geq 1$, if we have

$$\text{MOD}_q(x_1, \ldots, x_n) = \dot{\bigvee}_{i \in [T]} C_i(x_1, \ldots, x_n),$$

where each C_i is an $\text{AC}^0_d[p]$ circuit, then

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Proof sketch.
Polynomial approximation method in the very low error regime.

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Polynomial approximation method in the very low error regime.

(Razborov/Smolensky's lower bound: optimized when $\varepsilon = \Omega(1)$.)

Improved approximation by $\mathbb{F}_p$ polynomials

Polynomial approximation method + Uniqueness:

Claim. If each C_i can be δ-approximated by an $\mathbb{F}_p$ polynomial P_i, then

$$Q(x) \stackrel{\text{def}}{=} \sum_{i \in [T]} P_i(x) \qquad \textbf{(Recall: } f = \dot{\bigvee}_{i \in [T]} C_i)$$

is an $\varepsilon = T \cdot \delta$ approximator for f.

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Reason.
In general, several P_i's correct on x can cause "$\bigvee$" to be wrong ($\mathbb{F}_p$).
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Problem: how to control error and degree simultaneously?

The low error regime in the approximation method

Razborov/Smolensky, 1987 (polynomial approximation)
For every $\delta(n) > 0$, any $\mathsf{AC}^0_d[p]$ admits a δ-error probabilistic polynomial $\mathbf{P}(x_1, \ldots, x_n) \in \mathbb{F}_p[x_1, \ldots, x_n]$ of degree $(O(\log n + \log(1/\delta)))^d$.

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Kopparty and Srinivasan, 2012 (extension)

$(O(\log n))^d \cdot \log(1/\delta)$ **instead of** $(O(\log n + \log(1/\delta)))^d$.

The low error regime in the approximation method

Razborov/Smolensky, 1987 (polynomial approximation)
For every $\delta(n) > 0$, any $\mathrm{AC}^0_d[p]$ admits a δ-error probabilistic polynomial $\mathbf{P}(x_1, \ldots, x_n) \in \mathbb{F}_p[x_1, \ldots, x_n]$ of degree $(O(\log n + \log(1/\delta)))^d$.

Kopparty and Srinivasan, 2012 (extension)

$(O(\log n))^d \cdot \log(1/\delta)$ **instead of** $(O(\log n + \log(1/\delta)))^d$**.**

Razborov/Smolensky + folklore, 1987 (lower bound for all ε)
For every $\varepsilon(n) \in [2^{-.001n}, 1/100q]$, any $Q(x_1, \ldots, x_n) \in \mathbb{F}_p[x_1, \ldots, x_n]$ that ε-approximates MOD_q (uniform distribution) has degree

$$\Omega\left(\sqrt{n \cdot \log(1/\varepsilon)}\right).$$

Finishing the proof

Suppose $\mathrm{MOD}_q(x_1, \ldots, x_n) = \dot{\bigvee}_{i \in [T]} C_i(x_1, \ldots, x_n)$.

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$$\begin{aligned} \text{degree} &\leq (\log n)^d \cdot \log(1/\delta) \\ &= (\log n)^d (\log T + \log(1/\varepsilon)). \end{aligned}$$

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which is maximized when $\varepsilon = \exp\left(- n/(4(\log n)^{2d}) \right)$.

Observation

To obtain $\mathrm{AC}^0_d[p]$ circuit size lower bounds for MOD_q:

Polynomial approximation method with ε as large as possible.

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Polynomial approximation method with ε as large as possible.

To understand structure of optimal polynomial size circuits up to depth $\approx \log n / \log\log n$:

Polynomial approximation method in the very low error regime.

Round complexity in $\mathcal{C}$-compression games

$AC^0[p]$ lower bound: holds for any number of rounds.

$AC^0[p]$ upper bound: single-round compression.

Power of interaction in compression games?

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Chattopadhyay and Santhanam, 2012:
For every fixed r, there is a Boolean function on n variables that admits AC^0-bounded protocols with r rounds and cost $O(n^{1/r})$, but for which any correct AC^0-bounded $(r-1)$-round protocol has cost $\Omega(n^{2/r-o(1)})$.

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$\Longrightarrow$ Quadratic gap, dependence on r not very satisfactory.

The power of interaction in AC^0-compression games

[Theorem 2].

Let $r \geq 2$ and $\varepsilon > 0$ be fixed parameters. There is an explicit family of functions $f = \{f_n\}_{n \in \mathbb{N}}$ with the following properties:

- There exists an $AC^0_2(n)$-bounded protocol Π_n for f_n with r rounds and cost $c(n) \leq n^{\varepsilon}$, for every $n \geq n_f$, where n_f is a fixed constant that depends on f.

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- Any $AC^0(\text{poly}(n))$-bounded protocol Π for f with $r - 1$ rounds has cost $c(n) \geq n^{1-\varepsilon}$, for every $n \geq n_\Pi$, where n_Π is a fixed constant that depends on Π.

Hard function for round-limited protocols

Function $f_n \colon \{0,1\}^n \to \{0,1\}$, where $n \stackrel{\text{def}}{=} m + \ell \cdot r \cdot m$.

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Intuition:

Upper bound: $r + 1$ rounds with communication $(1 + r) \cdot m$.
Lower bound: r rounds require communication at least $\ell \cdot m^{1-o(1)}$.

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Proof relies on a round elimination argument via random restrictions, together with an appropriate induction hypothesis.

Part 3: Open Problems

Open Problem 1: Round separation for $AC^0[p]$-compression games?

As far as we know, single-round $AC^0[p]$ protocols are as powerful as k-round protocols.

(Our technique for $AC^0[p]$ is insensitive to the # of rounds.)

Problem. Prove a "round separation theorem" for $AC^0[p]$-compression games.

Open Problem 2: Lower bounds for randomized $AC^0[p]$-compression games?

The randomized $AC^0[p]$-compression complexity of Majority remains open.

Reason: proof explores very low error regime in the polynomial approximation method (initial error probability is not tolerated).

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Remark. Communication cost is $n/(\log n)^{\Theta(d)}$ for randomized AC^0_d-compression games (Chattopadhyay and Santhanam, 2012).

Open Problem 3: Power of modulo m gates in interactive compression?

Unconditional lower bounds:

Circuit class	**Hard function**	**Incompressibility (depth d)**
AC^0	Parity	$CC(\text{Parity}_n) \geq n / \log^{O(d)} n$
$AC^0[p]$	Majority	$CC(\text{Majority}_n) \geq n / \log^{O(d)} n$
$AC^0[m]$	NEXP, Majority (?)	$CC(\text{Majority}_n) = ?$

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Question. Are there randomized $AC^0[m]$-compression games for Majority with communication cost $n^{1-\varepsilon}$?

This result would shed more light on the hardness of proving lower bounds against circuits with modulo m gates.

Thank you!