Unprovability of circuit upper bounds in Cook's theory PV

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– Based on joint work with **Jan Krajíček** (Prague).

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Motivation

Question. Is there $f \in \mathsf{P}$ such that f does not admit non-uniform circuits of size $O(n^k)$?

Natural candidates:

- The ℓ-clique problem on n-vertex graphs?
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As far as we know, every problem in P might admit linear size circuits.

Can we at least show that some formal theories cannot prove that $\mathsf{P} \subseteq \mathsf{SIZE}(n^k)$?

Previous Work

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The closest reference seems to be

S. Cook and J. Krajíček, "*Consequences of the provability of* $\mathrm{NP} \subseteq \mathrm{P/poly}$", 2007.

where conditional independence results were obtained for the theories PV, S^1_2, and S^2_2.

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2. Discuss a theory (PV) that "understands" this sentence, and mention results that can be formulated and proved in PV.

3. Sketch the ideas behind the argument that PV cannot prove that P $\subseteq$ SIZE(n^k), formalized as in **1.** above.

4. Discussion and open problems.

1. Formalizing non-uniform circuit upper bounds

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- What is $\mathbb{N}$? What about $\{0,1\}^n$? A circuit? Symbol "$\in$"? Etc.

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$|z|^k$ means $|z| \times \ldots \times |z|$, etc. (we have function symbols $+$ and $\times$).

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- We need a theory that is connected to uniform polynomial time computations, and that can use first-order quantifiers.

2. The first-order theory PV

(informal discussion)

Background

- PV (“Polynomially Verifiable”) introduced as an equational theory by S. Cook in 1975:
  “Feasibly constructive proofs and the propositional calculus”.

Based on work of Cobham (1965) characterizing p-time functions by a function algebra.

Motivation: Formalizes feasible reasoning, connection to NP vs. coNP problem (propositional translations).

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- First-order formulation (PV_1) presented in Krajíček, Pudlák, Takeuti (1991) as a conservative extension of the equational theory.

Background, cont.

- Definition of PV is technical. Some details not particularly important in our argument.

- Indeed, our unprovability results extends to the theory containing all true (in $\mathbb{N}$) universal sentences in the vocabulary $\mathcal{L}_{\mathsf{PV}}$ of PV.

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We shall give a brief (and incomplete) introduction to PV on the next few slides.
(A formal treatment appears in Section 5.3 of Krajíček's red book.)

An essentially equivalent formulation of the theory (perhaps more accessible) appears in:

E. Jeřábek, *The strength of sharply bounded induction*, 2006.

PV and its vocabulary $\mathcal{L}_{\mathsf{PV}}$

- Intended structure interpreting the symbols of PV is $\mathbb{N}$, together with p-time functions $\tilde{f}\colon \mathbb{N}^\ell \to \mathbb{N}$ interpreting each function symbol ("p-time algorithm") $f \in \mathcal{L}_{\mathsf{PV}}$.

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- The function symbols in $\mathcal{L}_{\mathsf{PV}}$ and (part of) the axioms of PV are introduced simultaneously, based on Cobham's characterization of FP.

PV and its vocabulary $\mathcal{L}_{\mathrm{PV}}$: Cobham's Theorem (1965)

Cobham's Theorem. FP is equivalent to the set of functions in $\mathbb{N}^k \to \mathbb{N}$, $k \geq 1$, obtained from the base functions below by composition and limited iteration on notation.

Base functions. 0, S, $\lfloor \frac{x}{2} \rfloor$, $2x$, $x \leq y$, $\mathrm{Choice}(x, y, z)$. (i.e. simple AC^0 functions)

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Limited iteration on notation.

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\begin{aligned}
f(\vec{x}, 0) &= g(\vec{x}) \\
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$$\begin{aligned} f(\vec{x}, 0) &= g(\vec{x}) \\ f(\vec{x}, y) &= h(\vec{x}, y, f(\vec{x}, \lfloor \frac{y}{2} \rfloor)), \end{aligned}$$

provided that $|f(\vec{x}, y)| \leq q(|\vec{x}|, |y|)$ for a fixed polynomial q and for all $\vec{x}, y \in \mathbb{N}$, where $|x| \stackrel{\text{def}}{=} \lceil \log(x + 1) \rceil$ is the length of the binary representation of x.

PV and its vocabulary $\mathcal{L}_{\mathsf{PV}}$, cont.

- As a new algorithm f is defined from previous ones:
  - We add a new function symbol f to $\mathcal{L}_{\mathsf{PV}}$,
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- We use first-order predicate calculus to reason and prove theorems in PV.

Remark. PV can be axiomatized by universal formulas (i.e., $\forall \vec{w}\ \phi(\vec{w})$, where ϕ is quantifier-free).

$\mathsf{UP}_{k,c}(f)$ as a sentence in $\mathcal{L}_{\mathsf{PV}}$

Recall $\mathsf{UP}_{k,c}(f)$:

$$\forall z\ \exists C\ \forall x\ \Big[\mathsf{Circuit}(C) \wedge \mathsf{size}(C) \leq c|z|^k \wedge \Big(|x| = |z| \rightarrow (f(x) \neq 0 \leftrightarrow \mathsf{CircEval}(C, x) = 1)\Big)\Big].$$

- Circuit(·), size(·), CircEval(·), etc. are poly-time algorithms which can be associated to well-behaved function symbols in $\mathcal{L}_{\mathsf{PV}}$.

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► Circuit($\cdot$), size($\cdot$), CircEval($\cdot$), etc. are poly-time algorithms which can be associated to well-behaved function symbols in $\mathcal{L}_{\mathsf{PV}}$.

Question. Given $k \geq 1$, is there a function symbol $h \in \mathcal{L}_{\mathsf{PV}}$ such that

$$\mathsf{PV} \nvdash \mathsf{UP}_{k,c}(h) \quad ? \quad \text{(no matter the choice of c)}$$

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$$\mathsf{PV} \nvdash \mathsf{UP}_{k,c}(h) \;\;?$$ (no matter the choice of c)

(By construction, the definition of $h \in \mathcal{L}_{\mathsf{PV}}$ contains in its description the specification of a poly-time algorithm for h.)

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- This often involves clever adaptations of the original arguments, approximations of probabilistic statements, discovering alternative proofs, etc.

The strength of PV, cont.

- A recent substantial formalization obtained in PV:

J. Pich, "*Logical strength of complexity theory and a formalization of the PCP Theorem in Bounded Arithmetic*, 2015.

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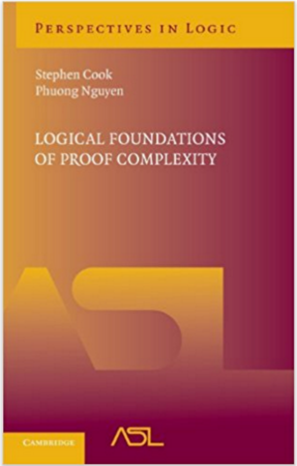
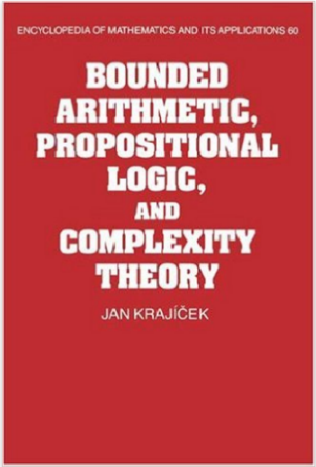
- Includes formalization of many other results, such as the Cook-Levin Theorem, expander graphs, etc.

For more information and background:





3. The unconditional unprovability result

(main ideas and the associated difficulties)

Main Theorem

Theorem. For every $k \geq 1$ there is a unary PV function symbol h such that for no constant $c \geq 1$ PV proves the sentence $\mathrm{UP}_{k,c}(h)$.

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Remark. $\mathrm{UP}_{k,c}(h)$ is a $\forall\exists\forall$-sentence in $\mathcal{L}_{\mathrm{PV}}$, and can be written as:

$$\mathrm{UP}_{k,c}(h) \;\equiv\; \forall z\, \exists C\, \forall x\; \phi_h(z, C, x), \text{ where } \phi_h \text{ is quantifier-free.}$$

The basic idea

- Logic/Provability as a bridge between non-uniform and uniform circuit complexity.

If PV $\vdash$ $\mathrm{UP}_{k,c}(h)$ using a proof π (list of symbols), extract from π computational information about sequence C_n of circuits computing h.

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(We will later explain why this natural approach is problematic.)

Techniques

Standard tools from logic and complexity, which build on other important results:

- Uniform circuit lower bounds (Santhanam-Williams, 2014).
- Formalization of the argument from Santhanam-Williams in PV.
- Axiomatization of PV as a universal theory.
- Herbrand's Theorem from mathematical logic.
- Krajicek-Pudlak-Takeuti Theorem (KPT) from bounded arithmetic.
- (Non-constructive) Inductive argument.

The uniform lower bound

R. Santhanam and R. Williams, "*On uniformity and circuit lower bounds*", 2014.

Theorem. For every $k \geq 1$, there is $L \in \mathsf{P}$ such that $L \notin$ P-uniform-SIZE(n^k).

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$L \in \mathrm{DTIME}(n^\ell)$, but P-uniform generating algorithm can run in time n^{2^ℓ}, $n^{2^{2^{\ell \cdot k}}}$, etc.

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► **Our Approach.** From a PV-proof of $\mathrm{UP}_{k,c}(h)$, we try to extract a poly-time generating algorithm. We can't control its p-time bound, but this is okay with the theorem above!

The KPT Witnessing Theorem

J. Krajíček, P. Pudlák, and G. Takeuti: "*Bounded arithmetic and the polynomial hierarchy*", 1991.

Theorem. Assume T is a universal theory with vocabulary $\mathcal{L}$, ϕ is a quantifier-free $\mathcal{L}$-formula, and

$$T \vdash \forall z \, \exists C \, \forall x \;\; \phi(z, C, x) \; .$$

Then there exist a constant $d \geq 1$ and a finite sequence $t_1, \ldots, t_d$ of $\mathcal{L}$-terms such that

$$T \vdash \phi(z, t_1(z), x_1) \; \vee \; \phi(z, t_2(z, x_1), x_2) \; \vee \; \ldots \; \vee \; \phi(z, t_d(z, x_1, \ldots, x_{d-1}), x_d).$$

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- The result can be established using proof theory or model theory.

Applying the KPT Theorem to PV and $\mathrm{UP}_{k,c}(f)$

- Fix $k \geq 1$, and assume that for every $f \in \mathcal{L}_{\mathrm{PV}}$ we have $c \geq 1$ such that

$$\mathrm{PV} \vdash \mathrm{UP}_{k,c}(f) \qquad \text{Recall that this is } \forall z\, \exists C\, \forall x\, \phi_f(z, C, x).$$

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Applying the KPT Theorem to PV and $\mathrm{UP}_{k,c}(f)$

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The general case

- If $d > 1$, we obtain from $\mathsf{PV} \vdash \mathsf{UP}_{k,c}(f)$ the more general scenario:

$$\mathbb{N} \models \phi(z, t_1(z), x_1) \lor \phi(z, t_2(z, x_1), x_2) \lor \ \ldots \ \lor \phi(z, t_d(z, x_1, \ldots, x_{d-1}), x_d).$$

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- **Contradiction? A difficulty is the lack of super-linear non-uniform lower bounds!**

How to establish an unconditional unprovability result?

- Apply KPT to a specific $\mathsf{UP}_{k,c}(g)$, obtaining a disjunction of $\leq d$ formulas, $d \in \mathbb{N}$.
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Check our paper for more details!

4. Remarks and open problems

Consistency of lower bounds

Given k and a "hard" $h \in \mathcal{L}_{\mathsf{PV}}$, by a standard compactness argument over the formulas

$$\mathsf{PV} \ \cup \ \{\neg \mathsf{UP}_{c,k}(h) \mid c \in \mathbb{N}\},$$

Corollary. For every $k \geq 1$ there exists a unary PV function symbol h and a model $\mathfrak{M}_k$ of PV such that for every $c \geq 1$,

$$\mathfrak{M}_k \ \models \ \neg UP_{k,c}(h).$$

Consistency of lower bounds, cont.

- From the point of view of the structure $\mathfrak{M}_k$, there are poly-time computations that require non-uniform circuits of size $\omega(n^k)$.

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(In the spirit, for instance, of ZF Set Theory and the consistency of the Axiom of Choice.)

Open problems and directions

- Prove a similar independence result for theories stronger than PV.

Example: $\mathsf{APC}_1 \stackrel{\text{def}}{=} \mathsf{PV} + \mathsf{dWPHP}(L_{\mathsf{PV}})$, a theory that formalizes many probabilistic arguments and randomized algorithms (Jeřábek's phd thesis, 2005), including:

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- Obtain an explicit function symbol h in our result (instead of only an existential proof).

- Establish the same unprovability result under a $\exists\forall\exists\forall$-formalization of the upper bound statement, which also quantifies over the parameter c in the size bound cn^k.

Thank you.