On the Extension Complexity of Polytopes Associated with Permutation Groups

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PART I

POLYTOPES vs GROUPS
Popular Optimization Problem

\[
\text{Max } \sum_{i} c_i x_i \quad \text{ over } \quad \text{A discrete domain } \Sigma
\]

\[
\hat{x} \in \mathcal{F} \subseteq \Sigma^n
\]

\[
\text{Set of feasible solutions}
\]
POPULAR APPROACH

IDENTIFY POINTS IN $F \subseteq \Sigma^n$
WITH A SET $\hat{F}$ OF VECTORS IN $\mathbb{R}^n$

$$P(F) = \text{conv}(\hat{F})$$

OPTIMIZE $\sum_{\hat{a}} c_{\hat{a}} x_{\hat{a}}$
OVER $P(F)$
ISSUE: THE POLYTOPE P(F) MAY HAVE SUPERPOLYNOMIALY MANY FACETS.

DESCRIPTION OF P(F) MAY REQUIRE SUPERPOLYNOMIALY MANY INEQUALITIES
WORK AROUND: EXTENDED FORMULATIONS

\[ Q \subseteq \mathbb{R}^{n + n'} \quad n' = \text{POLY}(n) \]

MAXIMIZE \( \sum c_i x_i \) HERE

GET A SOLUTION HERE
EXAMPLE: PERMUTAHEDRON

\[ P_N = \{ a_1, a_2, \ldots, a_m \in [N]^N : a_i \neq a_j \text{ for } i \neq j \} \]

\[ P(P_N) = \]

\[ P(P_N) \text{ has } 2^{\Omega(N)} \text{ facets} \]

But \( \Theta(N \log N) \) ext. form.  

Image from Wikipedia
WHICH GROUPS HAVE POLYNOMIAL EXTENDED FORMULATIONS

\[ G \leq \text{SYM}(\{N\}) \quad \forall G \in G \Rightarrow \text{REPRESENT } G \text{ BY } a_1, a_2, \ldots, a_N \in \{N\}^N \]

\[ \hat{G} = \{ (a_1, a_2, \ldots, a_N) \in \mathbb{R}^N : a_1, a_2, \ldots, a_N \in G \} \]

\[ \text{G-HEDRON: } \mathcal{P}(G) = \text{CONV}(\hat{G}) \]

\[ \text{EX. } \mathcal{P}(\text{ALT}_n) \text{ IS KNOWN AS ALTERNAHEDRON} \]

\[ \checkmark \text{ ALSO HAS } \Theta(N \log N) - \text{SIZE EXT. FORM. (WELTGE-2012)} \]
Which groups have polynomial extended formulations

Sym$_n$, Alt$_n$ (Goemans, Weltge)

Reflection groups (Kaibel, Pasikovich, Humphreys)

Special case of main theorem of this work.

Thm: $X$ is a graph of treewidth $k$ and max degree $\Delta$ \Rightarrow

\begin{align*}
\chi_c(\text{Aut}(X)) &= 2^{O(k \Delta \log \Delta)} \cdot \chi(1) \\
&= 2^{O(k \Delta \log \Delta)} \cdot 1 \chi
\end{align*}
PART II

GROUPS vs GRAPHS
Let $[N] = \{1, \ldots, N\}$.

A group $G \subseteq \text{Sym}([N])$ is embeddable in a connected graph $X$ with $M \geq N$ vertices if...

\[ \text{Aut}(X) \text{ stabilizes } [N] = \{1, \ldots, N\} \]

\[ G = \{ \varphi|_N : \varphi \in \text{Aut}(X) \} \]

Graph embeddability complexity: $\text{g.e.c.}(G)$

Minimum $M$ s.t. $\exists X$ on $M$ vertices s.t. $G \rightarrow X$
Ex: symmetric group $\text{SYM}([n])$

Cyclic group $\mathbb{C}_n$
BÁBÁI, BOWER 1969:

ANY SUBGROUP \( G \subseteq \text{SYM}([N]) \) CAN BE EMBEDDED IN A GRAPH WITH \( O(N|G|) \) VERTICES.

THEOREM: WORST CASE FOR G.E.C. OF SUBGROUPS OF \( \text{SYM}([N]) \) IS \( O(N!) \).

OPEN PROBLEM: GIVE A FAMILY OF GROUPS WITH SUPERPOLYNOMIAL G.E.C.

BÁBÁI-1969: IS THE G.E.C. OF \( \text{ALT}_N \) SUPERPOLYNOMIAL IN \( N \)?
A related problem:

Given a group $G$, find a graph $X$ such that $G \cong \text{Aut}(X)$.

Frucht-1949: Any group is isomorphic to the automorphism group of a 3-regular graph.

Contrast with: For $n \geq 6$, $\text{Sym}([n])$ cannot be embedded in graphs of max degree less than $n-1$.

Liebeck-1983: $\text{Alt}_n \cong \text{Aut}(X) \Rightarrow |X| \geq 2^{\Omega(n)}$.

Contrast with: Open whether $g.r.c.$ of $\text{Alt}_n$ is superpolynomial.
Which groups have polynomial extended formulations

Main Theorem:

If $G \in \text{SYM}([n])$ is embeddable in a graph $X$ with $m > n$ vertices, treewidth $k$ and max-degree $\Delta$ then

$$x.c. (G) = 2^0 \left( k \Delta \log \Delta \right) \cdot o(x).$$
PART III

FORMAL LANGUAGES

VS

POLYTOPES AND GROUPS
Which formal languages have small extension complexity?

By language we mean

\[ L_1, L_2, \ldots, L_N, \ldots \quad L_N \in (\Sigma_N)^n \]

we allow the alphabet to grow with the size of strings.
WHICH FORMAL LANGUAGES HAVE
SMALL EXTENSION COMPLEXITY?

(EXPLICIT IN Tiwary 2015)

IF \( L_n \) IS COMPUTABLE BY NON. UNIF. NONDET.
READ-ONCE OBLIVIOUS BRANCHING PROGRAMS
OF SIZE \( s(n) \) THEN \( x.\text{c}(L_n) \leq s(n)^{o(1)} \)

\[ \downarrow \]

ONLINE NONDET. TURING MACHINE WORKING IN
SPACE \( r(n) \Rightarrow x.\text{c}.(L_n) \leq 2^{o(r(n))} \cdot n \)

\[ \downarrow \]

REGULAR LANGUAGES \( \Rightarrow x.\text{c}.(L_n) \leq o(n) \)
Which formal languages have small extension complexity?

Open problem: Suppose $L_n = \sum^n \cap L(G)$ for some context-free grammar $G$. Is $x.c.(L_n) = N^f(|G|)$?

In other words do context-free languages have polynomial $x.c.$?
EX: REGULAR CFG'S ARE 1-HOMOGENEOUS.
(PARSE TREES ARE LINES)

ANY LANGUAGE CAN BE ACCEPTED
BY NON UNIF. FAMILIES OF 1-HOMOG.
CFG'S.

PALINDROME LANGUAGE IS ACCEPTED
BY CONST. SIZE 1-HOMOGENEOUS CFG.

\[
G: \begin{align*}
A & \rightarrow 112 \\
A & \rightarrow 1A1 \\
A & \rightarrow 2A2
\end{align*}
\]

EVERY WORD OF SIZE N IS PARSED
BY TREES OF SAME SHAPE.
$h$-HOMOGENEOUS CFG's

Say that a CFG is $h$-homogeneous if for each $n$, the strings of length $n$ are accepted using one out of $h(n)$ parse-trees.
THEOREM: IF $L_n$ IS ACCEPTED BY A $h$-HOMOGENEOUS CFG $G$ THEN
$$x \cdot c(D(L_n)) \leq h(n) \cdot |G|^{O(1)} \cdot n^{O(1)}.$$ 

THEOREM: IF $G \in \text{SYM}([n])$ CAN BE EMBEDDED ON A GRAPH OF SIZE $M$, TREewidth $k$ AND MAX-DEGREE $\Delta$ THEN THERE IS A 1-HOMOGENEOUS CFG $G$ OF SIZE $O(k \cdot \Delta \log \Delta) \cdot M^{O(1)}$ S.T. $L(G) = G$. 
OPEN PROBLEMS

1) Is \( g.e.c(\text{ALT}_n) \gtrsim N^{\Omega(1)} \)?

Note that \( \text{x.c.}(\text{P(ALT}_n)) = \Theta(N \log N) \)

2) Is \( \text{x.c.}(\mathbb{Z}^n) \leq N^f(|G|) \) for some computable function \( f \)?

3) Give a group with superpolynomial \( x.c. \).

4) Give a group with superpolynomial \( g.e.c. \).
THANK YOU!