Extractor-Based Time-Space Lower Bounds for Learning

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Joint work with Ran Raz and Avishay Tal
[Shamir 2014], [Steinhardt-Valiant-Wager 2015]

Initiated a study of memory-samples lower bounds for learning

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(when the samples are viewed one by one)

(also known as online learning)
Example: Parity Learning

$x \in \mathbb{R} \{0,1\}^n$ is unknown
A learner tries to learn $x$ from a stream $(a_1, b_1), (a_2, b_2) \ldots$, where $\forall t$:

$a_t \in \mathbb{R} \{0,1\}^n$ and

$b_t = a_t \cdot x$ (inner product mod 2)
Example: Parity Learning

\[ x \in \mathbb{R} \{0,1\}^n \] is unknown

A learner tries to learn \( x \) from a stream

\((a_1, b_1), (a_2, b_2), \ldots\), where \( \forall t : a_t \in \mathbb{R} \{0,1\}^n \) and

\[ b_t = a_t \cdot x \] (inner product mod 2)

In other words:

We get random linear equations in \( x_1, \ldots, x_n \), one by one, and need to solve them

(no noise)
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By solving linear equations:

\( O(n) \) samples, \( O(n^2) \) memory bits

By trying all possibilities:

\( O(n) \) memory bits, exponential number of samples
Raz’s Breakthrough [2016]

Any algorithm for parity learning requires either $\Omega(n^2)$ memory bits or an exponential number of samples

Conjectured by:

Steinhardt, Valiant and Wager [2015]
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[Kol-Raz-Tal 2017]:

Any algorithm for learning sparse parities (hence also: DNF, CNF, decision tress, Juntas) requires either super-linear memory size or a super-polynomial number of samples
For a large class of learning problems, any learning algorithm requires either quadratic memory size or an exponential number of samples.

A new and general proof technique

[Raz 2017]
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A new and general proof technique

As a special case: a new proof for the memory-samples lower bound for parity learning.

Our result uses a similar proof technique.

[Raz 2017]
Other Related Work

Independently, [Moshkovitz, Moshkovitz 2017a]: Any algorithm requires either $\sim 1.25n$ memory bits or an exponential number of samples for large class of learning problems.

Subsequently, [Moshkovitz, Moshkovitz 2017b]: Similar results as [Raz 2017]
Other Related Work

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Independently of our result, [Beame, Oveis-Gharan, Yang 2018] proved related lower bounds.
Motivation

Learning Theory: [S 14, SVW 15,...] In some cases, learning is infeasible, due to memory constraints
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Cryptography: [R 16, VV 16, KRT 16] Bounded Storage Crypto - Key’s length: \( n \)  Encryption/Decryption time: \( n \)

Unconditional security, if the attacker’s memory size is at most \( \frac{n^2}{25} \)
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Complexity Theory: Different Time-Space Tradeoffs have been studied in many models [BJS 98, Ajt 99, BSSV 00, For 97, FLvMV 05, Wil 06,...]
A Learning Problem as a Matrix

\( A, X \) : finite sets

\( M: A \times X \rightarrow \{-1, 1\} : \) a matrix
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\(x \in_R X\) is unknown. A learner tries to learn \(x\) from a stream \((a_1, b_1), (a_2, b_2) \ldots\), where \(\forall t:\)

\(a_t \in_R A\) and

\(b_t = M(a_t, x)\)
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\( X : \) concept class = \( \{0, 1\}^n \)

\( A : \) possible samples = \( \{0, 1\}^{n'} \)
Our Result

Assume that any submatrix of $M$ of fraction $2^{-k} \times 2^{-\ell}$ has bias of at most $2^{-r}$. Then:
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Any algorithm requires either $\Omega(k \cdot \ell)$ memory bits or $2^{\Omega(r)}$ samples (Implies all previous results)
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(Implicit all previous results)

[R 17] looked only at largest singular value of $M$

An independent related result by [Beame, Oveis-Gharan, Yang 2018]
Our Result and [Beame, Oveis-Gharan, Yang 2018]

For large classes of learning problems, any learning algorithm requires either memory of size $\Omega((\log |A|) \cdot (\log |X|))$ or an exponential number of samples.

[R 17]: bound on memory of at most $\min((\log |A|)^2, (\log |X|)^2)$
Applications

Parity Learning: A learner tries to learn $x = (x_1, \ldots, x_n) \in \{0,1\}^n$, from random linear equations over $\mathbb{F}_2$.

$\Omega(n^2)$ memory or $2^{\Omega(n)}$ samples

Sparse Parities: A learner tries to learn $x = (x_1, \ldots, x_n) \in \{0,1\}^n$ of sparsity $l$, from random linear equations over $\mathbb{F}_2$.

$\Omega(n \cdot l)$ memory or $2^{\Omega(l)}$ samples
Applications

Learning from low-degree equations: A learner tries to learn $x = (x_1, ..., x_n) \in \{0,1\}^n$, from random multilinear polynomial equations of degree at most $d$, over $F_2$.

$\Omega(n^{d+1})$ memory or $2^{\Omega(n)}$ samples
Applications

Learning from low-degree equations: A learner tries to learn
\[ x = (x_1, \ldots, x_n) \in \{0,1\}^n \] , from random multilinear polynomial equations of degree at most \( d \) , over \( F_2 \).

\( \Omega(n^{d+1}) \) memory or \( 2^{\Omega(n)} \) samples

Low-degree polynomials: A learner tries to learn an \( n \)-variate multilinear polynomial \( p \) of degree at most \( d \) over \( F_2 \), from random evaluations of \( p \) over \( F_2^n \).

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And more..
Techniques to Prove Extractor Property

\[ M : A \times X \rightarrow \{-1, 1\} : \text{the learning matrix} \]

\[ M_a \text{ are } (\varepsilon, \delta)\text{-almost orthogonal} \]

For each row \( a_i \), at most \( \delta \) fraction of the rows \( a \in A \) have

\[ | < M_a, M_{a_i} > | \geq \varepsilon \]

Then, learning requires either \( \Omega(\log \frac{1}{\delta} \cdot \log (\min(\frac{1}{\varepsilon}, \frac{1}{\delta}))) \) memory or

\( \Omega(\min(\frac{1}{\varepsilon}, \frac{1}{\delta})) \) samples
Each layer represents a time step. Each vertex represents a memory state of the learner. Each non-leaf vertex has $2^{n'}+1$ outgoing edges, one for each $(a, b) \in \{0,1\}^{n'} \times \{-1,1\}$.
The samples \((a_1, b_1), \ldots, (a_m, b_m)\) define a computation-path. Each vertex \(v\) in the last layer is labeled by \(\hat{x}_v \in \{0,1\}^n\). The output is the label \(\hat{x}_v\) of the vertex reached by the path.
Proof Outline

$P_{x|v} =$ distribution of $x$ conditioned on the event that the computation-path reaches $v$

Significant vertices: $v$ s.t. $\|P_{x|v}\|_2 \geq 2^l \cdot 2^{-n}$
Proof Outline

\( P_{x|v} = \text{distribution of } x \text{ conditioned on the event that the computation-path reaches } v \)

Significant vertices: \( v \) s.t. \( \|P_{x|v}\|_2 \geq 2^l \cdot 2^{-n} \)

\( Pr(v) = \text{probability that the path reaches } v \)

We prove: If \( v \) is significant, \( Pr(v) \leq 2^{-\Omega(k \cdot l)} \)

Hence, there are at least \( 2^{\Omega(k \cdot l)} \) significant vertices
Proof Outline

If $s$ is significant, $Pr(s) \leq 2^{-\Omega(k\cdot l)}$

Progress Function: For layer $L_i$,

$$Z_i = \sum_{v \in L_i} Pr(v) \cdot \langle P_x|v, P_x|s \rangle^k$$

1) $Z_0 = 2^{-2n \cdot k}$

2) $Z_i$ is very slowly growing: $Z_0 \approx Z_m$

3) If $s \in L_m$, then $Z_m \geq Pr(s) \cdot 2^{2l \cdot k} \cdot 2^{-2n \cdot k}$

Hence: If $s$ is significant, $Pr(s) \leq 2^{-\Omega(k\cdot l)}$
Generalization to Non-Product Distributions

$A, X$ : finite sets

$P: A \times X \rightarrow [0,1]$ : a joint distribution

$x \in_R X$ is unknown

A learner tries to learn $x$ from a stream $a_1, a_2 \ldots$, where $\forall t$:

$a_t$ is drawn randomly according to $P_{A|X=x}$
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For example: what if we get only positive samples, large output...
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\( \tilde{M} : A \times X \rightarrow \mathbb{R} \): \( \tilde{M}(a, x) = \frac{P_{A|X=x}(a)}{P_A(a)} - 1 \)
Generalization to Non-Product Distributions

\( \tilde{M} : A \times X \to \mathbb{R} : \tilde{M}(a, x) = \frac{P_{A|X=x}(a)}{P_A(a)} - 1 \)

If \( \max_{a,x} \tilde{M}(a, x) \leq 2^p \) and assume that any submatrix of \( M \) of fractional weight \( 2^{-k} \times 2^{-\ell} \) has bias of at most \( 2^{-r} \). Then:
Generalization to Non-Product Distributions

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Any algorithm requires either \( \Omega\left(\frac{k \cdot \ell}{p}\right) \) memory bits or \( 2^{\Omega(r)} \) samples
Generalization to Non-Product Distributions

\[ \tilde{M} : A \times X \to R : \tilde{M}(a, x) = \frac{P_{A \mid X = x}(a)}{P_A(a)} - 1 \]

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Application: For any finite field \( F \), learning a string \( x \in F^n \) from random linear equations, requires either a memory of size \( \Omega(n^2 \log(F)) \), or an exponential number of equations.
Open Problems

Secret/Samples over Reals

Optimal tradeoffs for DNFs..

Read-k learning
Thank You 😊