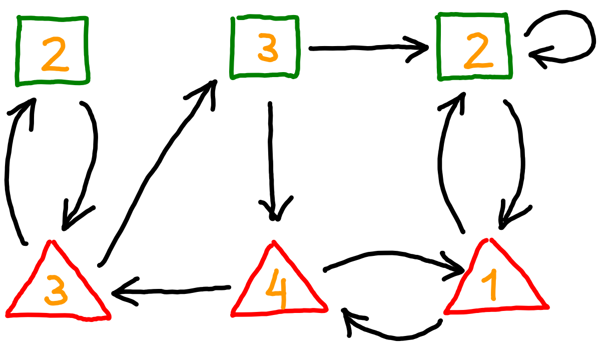
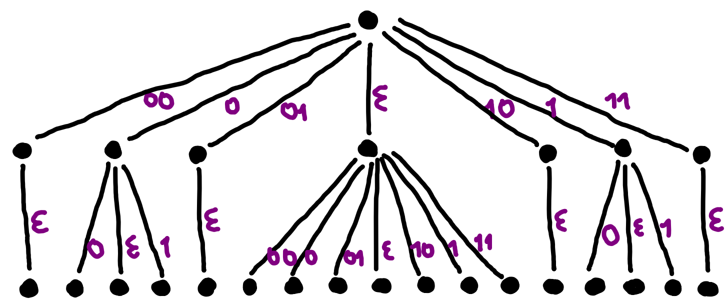
Parity Games and Universal Trees





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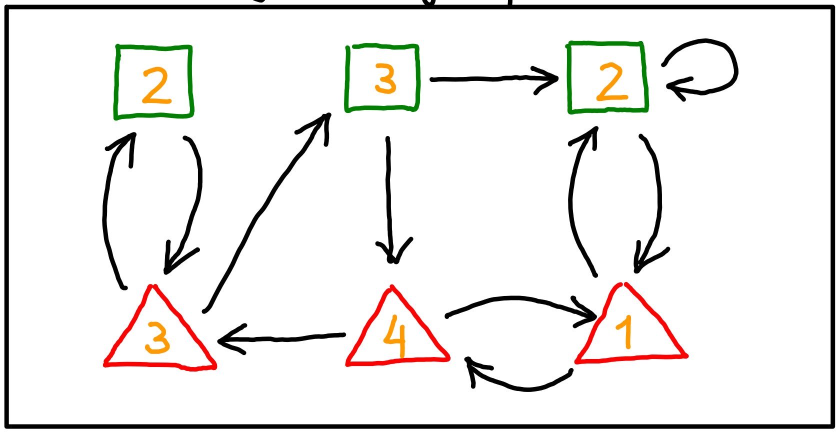
PARITY GAMES

$n = |V|$

A game graph



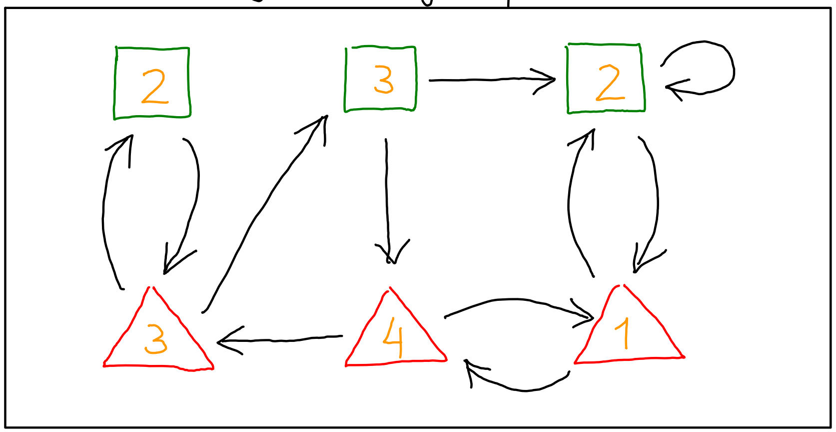
$G = (V = V_{Even} \uplus V_{Odd}, E, \pi)$

$\pi : V \to \{1, 2, 3, 4, 5, \ldots, d\}$

PARITY GAMES

$n = |V|$

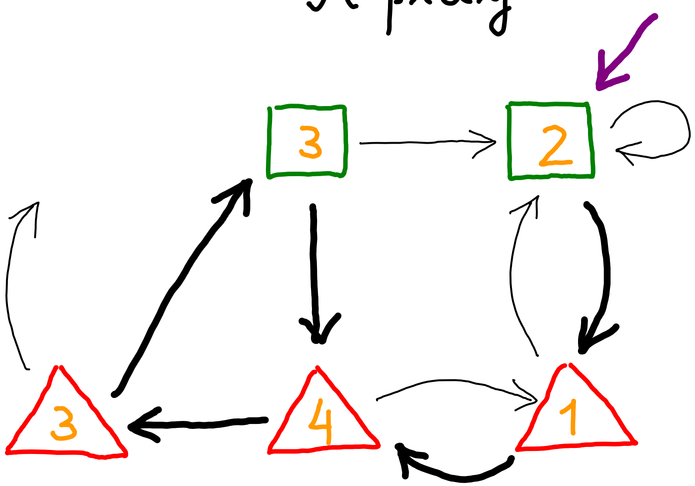
A game graph



$$G = (V = V_{Even} \uplus V_{Odd}, E, \pi)$$

$$\pi : V \to \{1, 2, 3, 4, 5, \ldots, d\}$$

A play



Even wins $\langle p_1, p_2, p_3, \ldots \rangle$

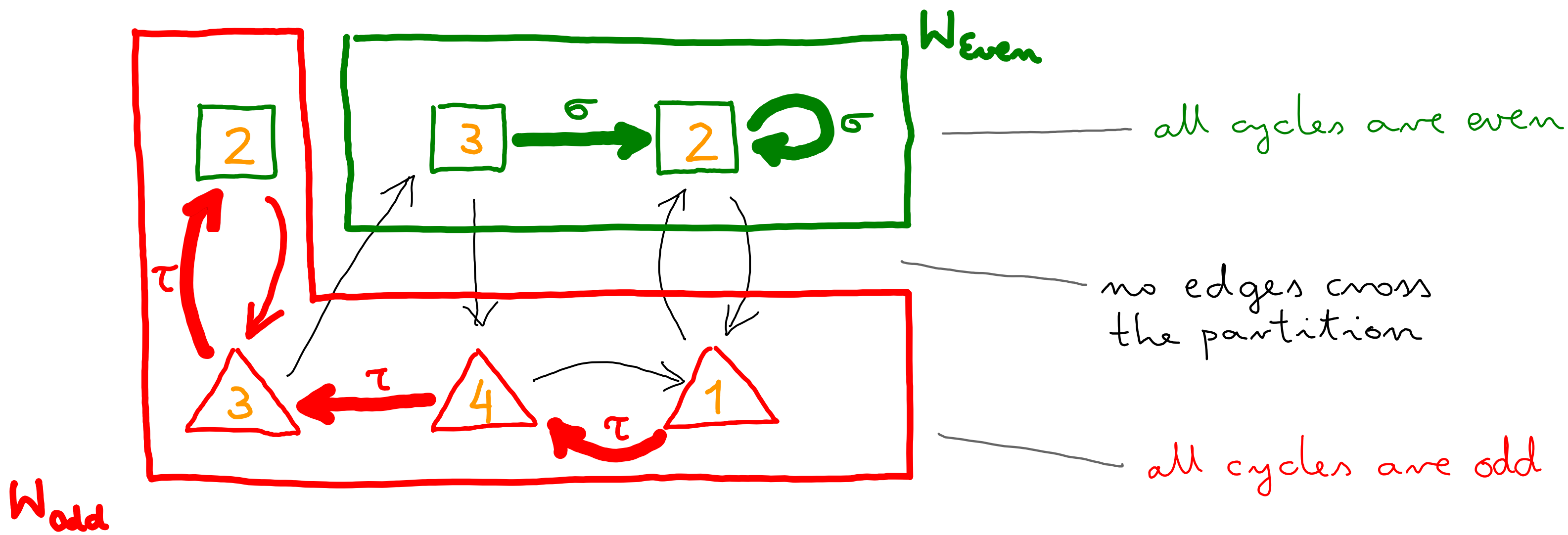
iff

$$\left[\limsup_{i \to \infty} p_i\right] \text{ is even}$$

SHORT WITNESSES FROM POSITIONAL DETERMINACY

THEOREM [Emerson, Jutla 1991; Mostowski 1991; re-proved since 1960's]

Parity games are positionally determined



COROLLARY [Emerson, Jutla, Sistla 1993]

Deciding the winner in parity games is $\boxed{\text{in } NP \cap co\text{-}NP}$

ALGORITHMS FOR SOLVING PARITY GAMES

- $\boxed{n^{d+O(1)}}$ [McNaughton 1993; Zielonka 1998]
- $\boxed{n^{\frac{d}{2}+O(1)}}$ [Browne, Clarke, Jha, Long, Marrero 1994; Seidl 1996; J. 2000]
- $\boxed{n^{d+O(1)}}$ strategy iteration [Vöge, J. 2000]

  $\boxed{2^{\Omega(n)}}$ [Friedmann 2009]

  - policy iteration for MDPs [Fearnley 2010]
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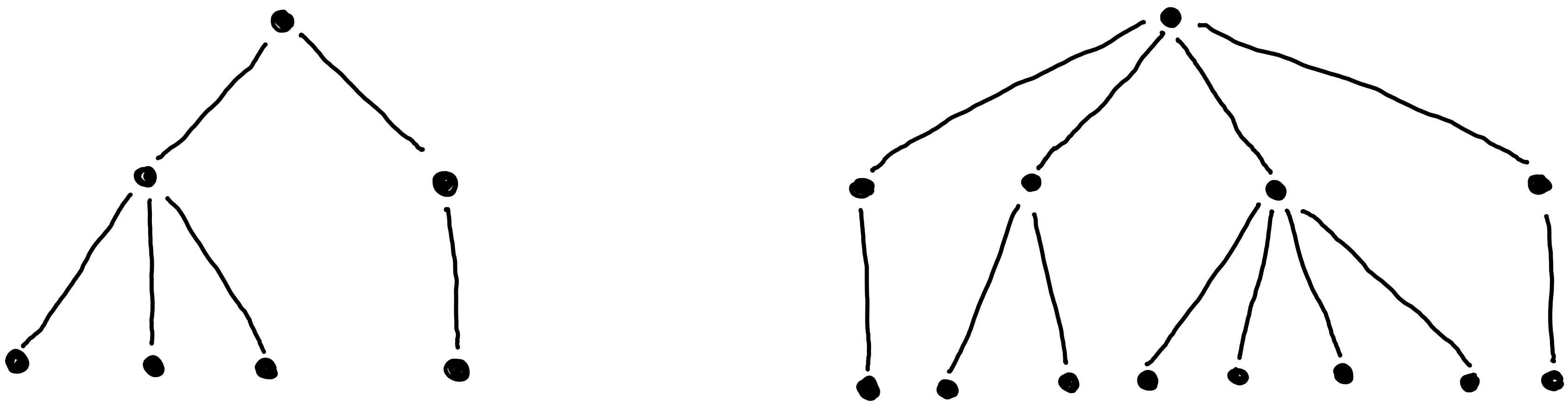
- $\boxed{n^{\lg d+O(1)}}$ **[Calude, Jain, Khoussainov, Li, Stephan 2017**; **J., Lazić 2017**; Lehtinen 2018; **Parys 2019**]

ORDERED TREES



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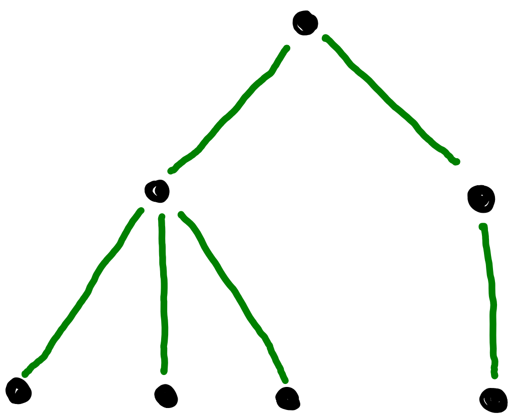
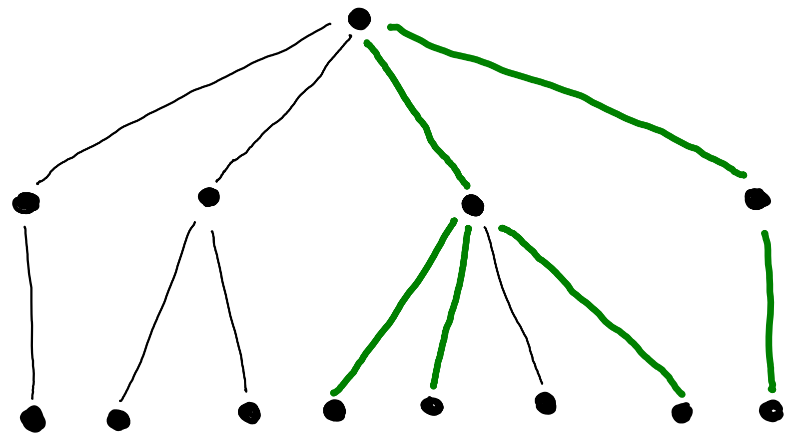


embeds into



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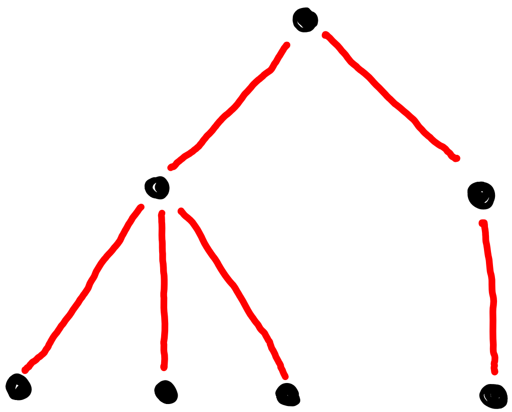
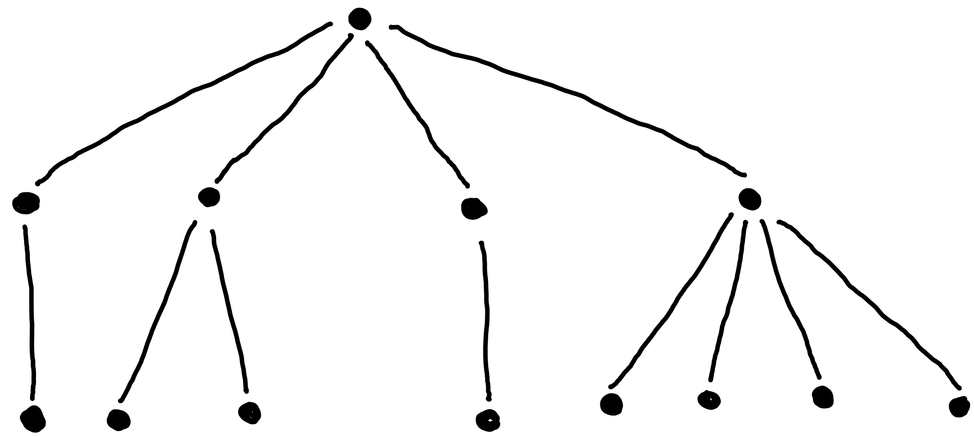
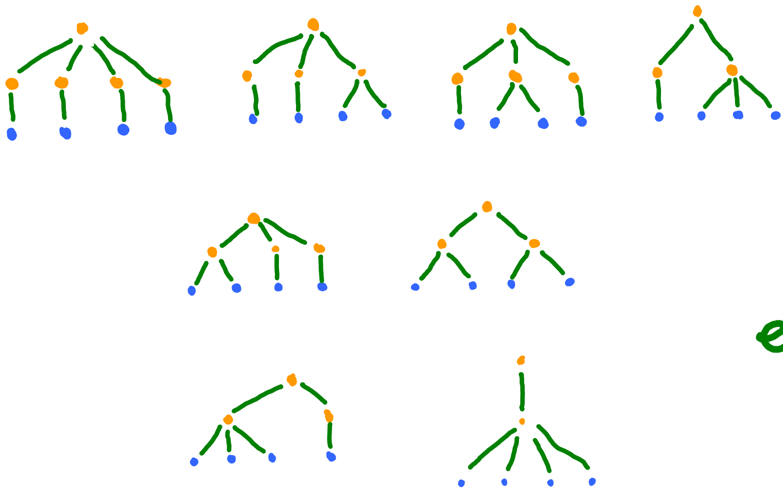


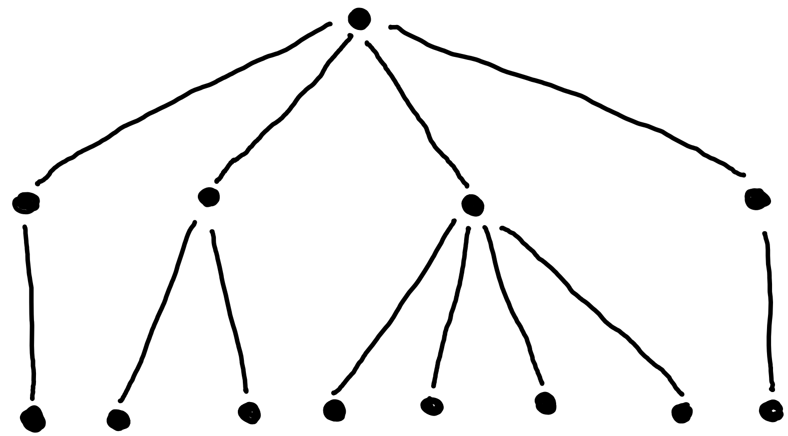
does not embed into



UNIVERSAL TREES



all embed into

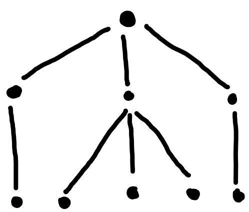


Leaves: 4
height: 2

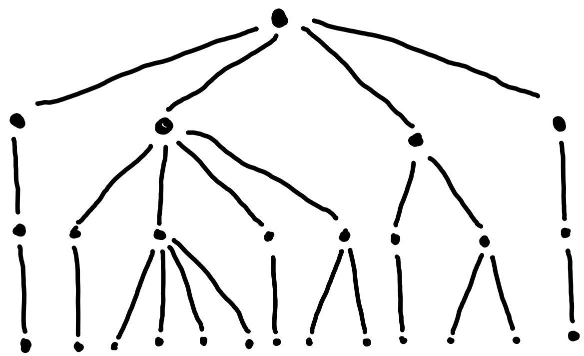
(4,2)-universal tree

UNIVERSAL TREES

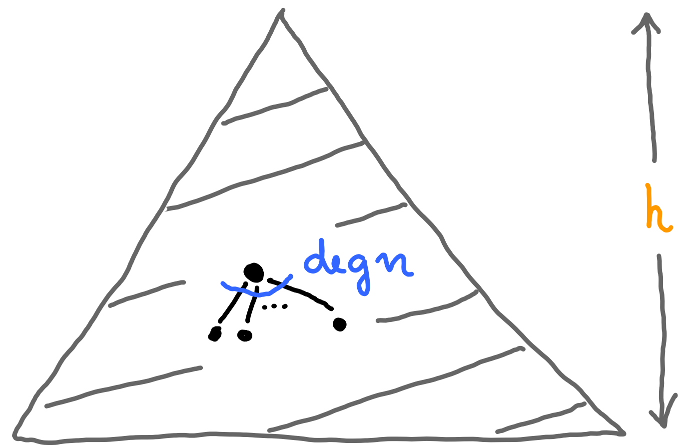
DEFINITION T is an (n,h)-universal tree if every tree with n leaves and of height h embeds into T



$(3,2)$-universal



$(4,3)$-universal



(n,h)-universal

$\Theta(n^h)$ size

UNIVERSAL TREES: A QUASI-POLYNOMIAL UPPER BOUND

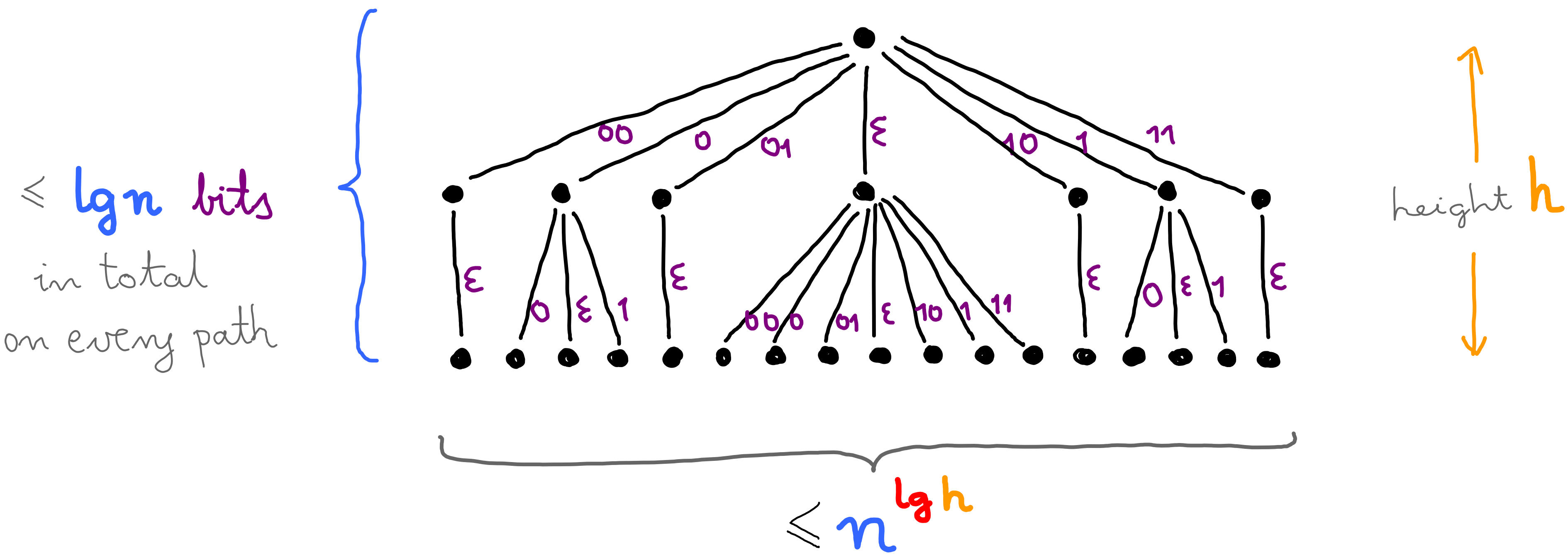
THEOREM [J., Lazić 2017]

There is an (n, h)-universal tree of size $n \cdot \binom{\lg n + h}{\lg n}$

- $\mathrm{poly}(n)$ if $h = O(\log n)$
- $n^{\lg\left(\frac{h}{\lg n}\right) + O(1)}$ if $h = \omega(\log n)$

UNIVERSAL TREES: A QUASI-POLYNOMIAL UPPER BOUND

(4,2)-universal ordered tree



$$\boxed{00 < 0 < 01 < \varepsilon < 10 < 1 < 11}$$

the linear order on bit strings induced by $0 < \varepsilon < 1$

The Strahler number

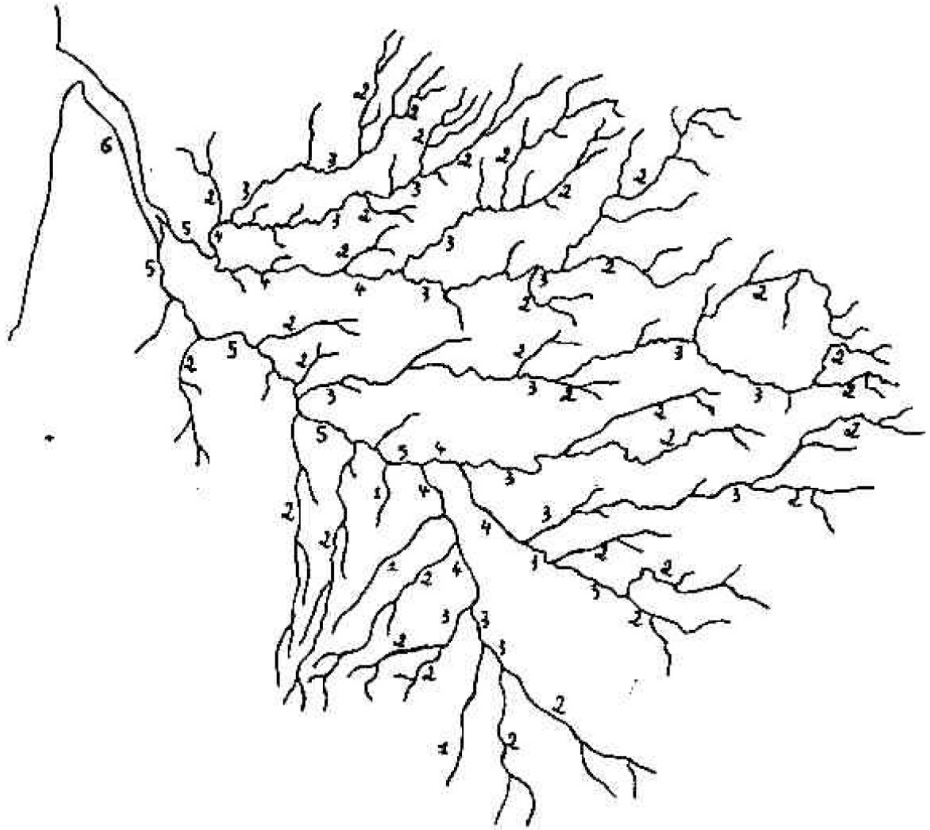


Fig. 2 : The Horton–Strahler rule for the Garonne river network

[Viennot 1990]

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- $\boxed{n^{SN(G)}}$ [Daviaud, J., **Thejaswini** 2019; J., **Morvan** 2019]