How Much Memory Is Needed to Win Infinite Games?[1]

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September 26, 1998

Abstract

We consider a class of infinite two-player games played on finitely coloured graphs. Our main question is: what is the exact size of memory needed by the players for their winning strategies. This problem is relevant to synthesis of reactive programs and to the theory of automata on infinite objects. Following ideas of Zielonka we provide matching upper and lower bounds for the size of memory needed by winning strategies in games with a fixed winning condition. We also show that in general the Latest Appearance Record data structure of Gurevich and Harrington is optimal.

1 Introduction...

...is still missing in this draft. Read the introduction in the conference version [DJW97] of this paper.

[1] This work was partially supported by Polish KBN grant No. 8 T11C 002 11.
[2] Partially supported by Polish-Danish Student Exchange Programme.
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2 Preliminaries

We consider infinite duration games played by two players called player 0 and player 1. Games are played on *arenas* that are directed graphs with vertices coloured by colours from a finite set of colours. More formally an arena is a tuple $\mathcal{A} = (V, V_0, V_1, E, C, \chi)$, where:

- V is a (possibly infinite) set of vertices,
- V_0, V_1 is a partition of V,
- $E \subseteq V^2$ is the set of edges,
- C is a finite set of colours,
- $\chi : V \rightharpoonup C$ is a partial function assigning colours to some vertices.

A *successor* of a vertex $v \in V$ is a vertex $w \in V$ such that $(v, w) \in E$. We assume that every vertex has at least one successor. Hence every finite path in the arena can be prolonged.

A (*Muller*) *game* is a pair $\mathcal{G} = (\mathcal{A}, \mathcal{F})$, where $\mathcal{A}$ is an arena and $\mathcal{F} \subseteq \mathcal{P}(C)$ is a *winning condition* (in Muller form [Tho90]). A *position* in the game is a finite path in $\mathcal{A}$. Note that even if the arena is finite (i.e., the arena has a finite number of vertices), the set of positions in the game is infinite, because in every arena there are finite paths of arbitrary length. The *initial position* of a *play* starting from a vertex v is the path consisting only of the vertex v. A *move* in a game consists of prolonging the current position by one vertex. During the play, if the last vertex in the current position is an element of V_0 then it is turn of player 0 to make a move, otherwise player 1 moves. The players make moves indefinitely. Note that there always exists a move to make, because we have assumed that every vertex has a successor. The *result of a play* in the game is an infinite path in the arena. For brevity, we often just say a play, meaning the result of the play.

The *winner* in a play $\pi = \langle v_0, v_1, v_2, \ldots \rangle$ is determined by referring to the winning condition $\mathcal{F}$. Let us define the set of *frequent colours* of π, denoted by $\mathit{Inf}(\pi)$, as the set of colours occuring infinitely often in the sequence $\langle \chi(v_0), \chi(v_1), \chi(v_2), \ldots \rangle$. A play π is a *winning play* for player 0 if the set of its frequent colours is an element of the winning condition, i.e., $\mathit{Inf}(\pi) \in \mathcal{F}$. Otherwise π is a winning play for player 1.

We say that a winning condition $\mathcal{F} \subseteq \mathcal{P}(C)$ has a *split* if there are sets $C_1, C_2 \in \mathcal{F}$, such that $C_1 \cup C_2 \notin \mathcal{F}$. A winning condition $\mathcal{F}$ is a *Rabin condition* if it does not have splits. It is a *Street condition* if $\mathcal{P}(C) \setminus \mathcal{F}$ does not have splits. And finally $\mathcal{F}$ is a *Mostowski condition* (or a *parity condition* [EJ91]) if it is both a Rabin and a Street condition.

Remark: In literature the notions of Rabin, Street and Mostowski conditions are usually defined in a different manner (see, e.g., the survey papers of Thomas [Tho90] and Niwiński [Niw97]). It can be, however, shown (see the paper of Zielonka [Zie94] for details) that our formulation is equivalent to the standard one. More precisely, a winning condition is a Rabin (Street, Mostowski) condition in our sense iff it *can* be represented in Rabin (Street, Mostowski) form in the usual sense. We say that a game $\mathcal{G} = (\mathcal{A}, \mathcal{F})$ is a Rabin (Street, Mostowski) game if the winning condition $\mathcal{F}$ is of the appropriate type.

A *strategy* for player 0 in a game $\mathcal{G}$ is a partial function $\zeta : V^* \rightharpoonup V_1$. This function may suggest only legal moves, i.e., for every $\pi = \langle v_0, v_1, \ldots, v_k \rangle$ the vertex $\zeta(\pi)$ must be a successor of v_k in the arena of the game. A play $\pi = \langle v_0, v_1, v_2, \ldots \rangle$ is *consistent* with a strategy ζ if $\zeta(v_0, \ldots, v_i) = v_{i+1}$ for every $i \in \mathbb{N}$ such that $v_i \in V_0$. A strategy ζ is *winning for player 0 from a vertex* v_0 if every play starting from v_0 consistent with ζ is winning for player 0. The *winning set* of player 0 is the set of vertices of the arena from which there exists a winning strategy for player 0. We say that a strategy is *winning for player 0 from a set* W, if it is winning for player 0 from every vertex $w \in W$. Strategies and the winning set for player 1 are defined similarly.

All the games we consider are *determined* [Mar75]. This means that from every vertex of the arena one of the players has a winning strategy. In other words, the winning sets for both players form a partition of the set of vertices. It turns out (see [GH82, McN93]) that for the games with Muller winning conditions the winning strategies can be realised by functions that do not have to refer to the whole current position of the game (recall that the set of positions is infinite), but only to a bounded amount of information about it. To make this precise we introduce the notion of a *strategy with memory.*

Definition 1 (Strategy with memory) Let $\mathcal{G}$ be a game as above and let M be a set called *memory.* A *strategy with memory* M (for player 0) is given by an element $m_0 \in M$ and a pair of functions: a *memory update function* $\zeta_M : M \times V \to M$, and a *next move function* $\zeta_V : V_0 \times M \to V_1$. The first function is used to determine the contents of the memory in a position of the game: $\zeta_M(v_0, \ldots, v_k) = \zeta_M(v_k, \zeta_M(v_{k-1}, \cdots \zeta_M(v_1, \zeta_M(v_0, m_0)) \cdots))$. The second function defines the strategy by $\zeta(v_0, \ldots, v_k) = \zeta_V(v_k, \zeta_M(v_0, \ldots, v_k))$.

We can think of a strategy with memory as an input/output automaton computing the strategy. This automaton inputs the moves taken by the opponent (player 1), keeps track of the memory in its finite control using the memory update function, and outputs the moves for the player (player 0) using the next move function.

Observe that if we take for M the set of all positions (i.e., all finite paths in the arena), and for ζ_M the identity function, then a strategy with memory is just a strategy as defined before. The notion of a strategy with memory is interesting in the case when the cardinality of M is smaller than the cardinality of the set of positions of the game. In particular, if we take M to be a one element set, then we obtain the notion of a *memoryless strategy.* Note that a memoryless strategy depends only on the current vertex. Hence one may view a memoryless strategy as a subgraph of the arena.

Remark: All strategies with memory we construct in this paper do not depend on the initial memory. So if a strategy given by some $m_0 \in M$ and (ζ_M, ζ_V)

is a winning strategy from some vertex v_0, then for every $m \in M$ the strategy given by m and (ζ_M, ζ_V) is a winning strategy from v_0 as well.

3 A generalization of LAR's

Gurevich and Harrington in their seminal paper [GH82] prove the *Forgetful Determinacy Theorem* for games with Muller winning conditions. They show that the *Latest Appearance Record* (LAR in short) data structure is sufficient as the memory for winning strategies in Muller games.

Thomas [Tho95] uses the LAR's and the *Memoryless Determinacy Theorem* for Mostowski games (due to Mostowski [Mos91] and Emerson and Jutla [EJ91]) to provide a reduction from Muller games to Mostowski games. This reduction gives both the Forgetful Determinacy Theorem for Muller games and an upper bound on the size of memory for winning strategies. The argument is, however, insensitive to the actual winning condition. No matter what the winning condition is, the blow-up of the reduction (and the memory size) is factorial in the number of colour used in the condition.

We design a generalisation of the LAR data structure and use it to provide a more succinct reduction of Muller games to Mostowski games. Following the ideas implicit in Zielonka's proof of the Forgetful Determinacy Theorem [Zie94] we introduce the notion of a *Zielonka tree* of a winning condition. The *generalised LAR's* for games with a winning condition $\mathcal{F}$ are just the leaves of the Zielonka tree of $\mathcal{F}$.

Notation: Let $\mathcal{F} \subseteq \mathcal{P}(C)$ be a winning condition. Define $\mathcal{F}\restriction D \subseteq \mathcal{P}(D)$ as the set $\{D' \in \mathcal{F} : D' \subseteq D\}$.

Definition 2 (Zielonka tree of a winning condition) We define the *Zielonka tree of* $\mathcal{F} \subseteq \mathcal{P}(C)$, denoted by $\mathcal{Z}_{\mathcal{F},C}$, inductively.

1. If $C \notin \mathcal{F}$ then $\mathcal{Z}_{\mathcal{F},C} = \mathcal{Z}_{\overline{\mathcal{F}},C}$ where $\overline{\mathcal{F}} = \mathcal{P}(C) \setminus \mathcal{F}$.
2. If $C \in \mathcal{F}$ then the root of $\mathcal{Z}_{\mathcal{F},C}$ is labelled with C. Let $C_0, C_1, \ldots, C_{k-1}$ be all the maximal sets in $\{X \notin \mathcal{F} : X \subseteq C\}$. Then we attach to the root, as its subtrees, the Zielonka trees of $\mathcal{F}\restriction C_i$, i.e. $\mathcal{Z}_{\mathcal{F}\restriction C_i, C_i}$, for $i = 0, 1, \ldots, k-1$.

Hence a Zielonka tree is a tree with nodes labelled by sets of colours. A node of $\mathcal{Z}_{\mathcal{F},C}$ is a *0-level node* if it is labelled with a set from $\mathcal{F}$. Otherwise it is a *1-level node*. In the sequel we will usually write $\mathcal{Z}_{\mathcal{F}}$ to denote $\mathcal{Z}_{\mathcal{F},C}$, if C is clear from the context.

Example: Let $C_n = \{t_1, \ldots, t_n, f_1, \ldots, f_n\}$, for every $n \in \mathbb{N}$. For a boolean function $h : \{0,1\}^n \to \{0,1\}$ define a winning condition $\mathcal{F}_n \subseteq \mathcal{P}(C_n)$ to be the family $\{H_1, \ldots, H_k\} = \big\{\{x_1, \ldots, x_n\} \subseteq C_n : h(\overline{x_1}, \ldots, \overline{x_n}) = 0 \text{ and } x_i \in \{t_i, f_i\}\big\}$, where $\overline{t_i} = 1$ and $\overline{f_i} = 0$ for every $i = 1, \ldots, n$. The root of the Zielonka tree $\mathcal{Z}_{\mathcal{F}_n}$ (c.f. Figure 1) is a 1-level node labelled with the set C_n of all

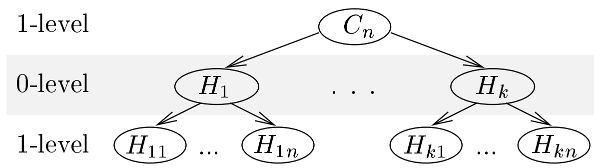


Figure 1: Example of a Zielonka tree

colours. The k children of the root are 0-level nodes labelled with $H_1, \dots, H_k$ respectively. Every node labelled with $H_i = \{h_{i1}, \dots, h_{in}\}$ has n children labelled with the sets $H_{i1}, \dots, H_{in}$ respectively, where $H_{ij} = H_i \setminus \{h_{ij}\}$. These children are the leaves of the tree. □

By $l_{\mathcal{F}}$ we denote the number of leaves of the Zielonka tree $\mathcal{Z}_{\mathcal{F}}$, i.e., the number of generalised LAR's for the winning condition $\mathcal{F}$. We now show that there are fewer generalised LAR's than the ususal LAR's, which gives evidence that our new reduction is an improvement over the construction by Thomas.

Fact 3 For every winning condition $\mathcal{F} \subseteq \mathcal{P}(C)$ we have $l_{\mathcal{F}} \leq |C|!$.

Proof
The fact follows from the observation that every leaf of a Zielonka tree is uniquely determined by the labels on the path leading from the root to this leaf. The labels on such a path form a decreasing sequence of subsets of the set C of colours. There are no more than $|C|!$ such sequences of labels. □

The following easy observation gives a simple method of checking if a winning condition is a Rabin (Street, Mostowski) condition by looking at the shape of its Zielonka tree.

Fact 4 A winning condition $\mathcal{F} \subseteq \mathcal{P}(C)$ has a split iff there is a 1-level node in the Zielonka tree $\mathcal{Z}_{\mathcal{F}}$ with more than one child.

We consider the reduction of Muller games to Mostowski games in the context of the computational *problem of determining the winning sets* in a game: given a game $\mathcal{G} = (\mathcal{A}, \mathcal{F})$ and a starting vertex v_0, check if player 0 has a winning strategy from v_0 in $\mathcal{G}$. We have made the decision of representing a Muller winning condition $\mathcal{F} \subseteq \mathcal{P}(C)$ by its Zielonka tree $\mathcal{Z}_{\mathcal{F}}$. Thus an instance of the problem of deciding the winner in a Muller game is a tuple $(\mathcal{A}, \mathcal{Z}_{\mathcal{F}}, v_0)$. It is the same with Mostowski games; we only demand that the winning condition $\mathcal{F}$ is a Mostowski condition.

Note that by Fact 4 the Zielonka tree of a Mostowski condition has a very simple structure, it consists just of a path to the only leaf. Hence a Mostowski condition has a very compact representation, which is easily made linear in the number of colours $|C|$.

Other representations of a winning condition are possible, e.g., simple enumeration of all elements of the set $\mathcal{F} \subseteq \mathcal{P}(C)$. It might seem that the Zielonka tree representation is more succinct, however there are winning conditions for which the opposite is the case. Hence, in fact, the two representations are not comparable in this sense.

Theorem 5
There is a polynomial time reduction from the problem of determining the winning sets in in Muller games (with winning conditions represented by their Zielonka trees) to the problem of determining the winning sets in Mostowski games.

Proof
Let $(\mathcal{A}, \mathcal{Z}_{\mathcal{F}})$ be a Muller game, where $\mathcal{A} = (V, V_0, V_1, E, C, \chi)$ and $\mathcal{F} \subseteq \mathcal{P}(C)$. We assume that $C \in \mathcal{F}$; if it is not the case then we change the roles of the players, i.e., we consider the game $\overline{\mathcal{G}} = (\mathcal{A}, \overline{\mathcal{F}})$, where $\overline{\mathcal{F}} = \mathcal{P}(C) \setminus \mathcal{F}$.

Assume we have some fixed linear order on the nodes of $\mathcal{Z}_{\mathcal{F}}$. This allows us to talk about the *smallest descendant* of a node, about the *first child* of a node (i.e., the smallest, in the ordering, child of the node), etc. Similarly, having some child of a node we can naturally define what is the *next child* of this node. Let L denote the set of leaves of $\mathcal{Z}_{\mathcal{F}}$. For a colour $c \in C$ and a leaf $l \in L$ we define the *support of c and l*, denoted by $S(c, l)$, to be the node t of $\mathcal{Z}_{\mathcal{F}}$ that is the closest ancestor of l with c in its label. In particular, if c belongs to the label of l, then $S(c, l) = l$.

Now we construct a Mostowski game $\mathcal{G}' = (\mathcal{A}', \mathcal{F}')$. Then we show that winning strategies in $\mathcal{G}$ can be transferred to $\mathcal{G}'$, and vice versa. The arena of $\mathcal{G}'$ is $\mathcal{A}' = (V \times L, V_0 \times L, V_1 \times L, E', \{0, 1, \ldots, d\}, \chi')$, where d is the depth of $\mathcal{Z}_{\mathcal{F}}$, and E' and χ' are defined below.

We let $\big((v_1, l_1), (v_2, l_2)\big) \in E'$ if $(v_1, v_2) \in E$ and l_2 is calculated from v_1 and l_1 in the following way:

(i) We find the support $t = S(\chi(v_1), l_1)$ of $\chi(v_1)$ and l_1.

(ii) If $t = l_1$ then $l_2 = l_1$.

(iii) If $t \neq l_1$ then we find the son t' of t on the path to l. Let t'' be the next son of t; or the first son of t if t' is the last son. We set l_2 to be the smallest (in the ordering fixed at the beginning of the proof) leaf that is a descendant of t'' (see Figure 2).

Figure 2: ...still missing: sorry about that!

We define $\chi'\big((v, l)\big)$ to be the depth of the support $S(\chi(v), l)$. The depth of a node in a tree is the distance of the node from the root. In particular, the depth of the root is 0. Finally, we define $\mathcal{F}' = \big\{\ D \subseteq \{0, 1, \ldots, d\} : \min(D) \text{ is even } \big\}$. Clearly $\mathcal{F}' \subseteq \{0, 1, \ldots, d\}$ is a Mostowski condition and $d \leq |C|$.

It is not hard to see that $\mathcal{A}'$ and $\mathcal{Z}_{\mathcal{F}'}$ can be constructed from $\mathcal{A}$ and $\mathcal{Z}_{\mathcal{F}}$ in polynomial time. It remains to show what is the correspondence between the winning sets in $\mathcal{G}$ and $\mathcal{G}'$.

We show that player 0 has a winning strategy from a node (v,l) in $\mathcal{G}'$ iff player 0 has a winning strategy from the node v in $\mathcal{G}$. First, observe that there is a graph homomorphism $h : \mathcal{A}' \to \mathcal{A}$, defined by $h\big((v,l)\big) = v$. This homomorphism is a covering, i.e., for every (v,l) and w, such that $(v,w) \in E$, there is a unique l' with $\big((v,l),(w,l')\big) \in E'$. The other important property of $\mathcal{G}'$ is the following.

Claim 5.1 A play $\pi' \in (V \times L)^\omega$ in $\mathcal{G}'$ is winning for player 0 iff the play $h(\pi') \in V^\omega$ is winning for player 0 in $\mathcal{G}$.

Before we prove the claim let us see how it implies the theorem. We need to show that player 0 has a winning strategy from a vertex (v,l) in $\mathcal{G}'$ iff she has a winning strategy from the vertex v in $\mathcal{G}$. First, because h is a homomorphism, every strategy σ' in $\mathcal{G}'$ uniquely determines the strategy $h(\sigma')$ in $\mathcal{G}$, such that if a play π' is consistent with σ', then the play $h(\pi')$ is consistent with $h(\sigma')$. Then, as h is a covering, every strategy σ in $\mathcal{G}$ uniquely determines a strategy σ' in $\mathcal{G}'$, such that a play π is consistent with σ iff a play π' satisfying $h(\pi') = \pi$ is consistent with σ'. These translations together with the claim give us what we need to complete the proof of correctness of the reduction.

It remains to prove the claim. Let $\pi' = \langle (v_0,l_0),(v_1,l_1),(v_2,l_2),\ldots\rangle$ be a play as in the claim. Define p to be the minimum number appearing infinitely often in the sequence of colours of vertices in π' (recall that the colours used in $\mathcal{G}'$ are the natural numbers $\{0,1,\ldots,d\}$). In other words, p is the number such that:

- $\chi'\big((v_i,l_i)\big) = p$ for infinitely many $i \in \mathbb{N}$, and
- there is some $i_0 \in \mathbb{N}$ with $\chi'\big((v_{i_0},l_{i_0})\big) = p$, such that $\chi'\big((v_j,l_j)\big) \geq p$ for all $j \geq i_0$.

We want to show that p is even iff the set, $\mathit{Inf}(h(\pi'))$, of frequent colours in the play $h(\pi')$ belongs to $\mathcal{F}$.

Let $t = S(\chi(v_{i_0}), l_{i_0})$ be the support of $\chi(v_{i_0})$ and l_{i_0}, and let D be the label of t. From the definition of E' it follows that all the leaves l_j, for $j \geq i_0$, are descendants of t. This implies that $\chi(v_j) \in D$ for all $j \geq i_0$, hence $\mathit{Inf}(h(\pi')) \subseteq D$.

Let $\tilde{D}$ be the label of a son of t. We show that $\mathit{Inf}(h(\pi')) \not\subseteq \tilde{D}$. Let $i \geq i_0$ be such that $\chi(v_i) \in \tilde{D}$. Find the smallest index $j > i$ with $\chi'\big((v_j,l_j)\big) = p$. By the definition of χ' we have that $\chi(v_j) \notin \tilde{D}$. We can find such j for every $i \geq i_0$, hence there are infinitely many indices j with $\chi(v_j) \notin \tilde{D}$.

Thus $\mathit{Inf}(h(\pi'))$ is not included in the label of any of the sons of t. As $\mathit{Inf}(h(\pi')) \subseteq D$, it immediately follows from the definition of the Zielonka tree that $\mathit{Inf}(h(\pi')) \in \mathcal{F}$ iff $D \in \mathcal{F}$. By the definition of χ', $D \in \mathcal{F}$ iff p is even. Hence altogether we have that $Inf(h(\pi')) \in \mathcal{F}$ iff $D \in \mathcal{F}$ iff p is even. This concludes the proof of the claim and the theorem. □

Using our reduction and the memoryless determinacy of Mostowski games we obtain.

Corollary 6 The problem of deciding the winner in games with Muller winning conditions represented by their Zielonka trees is in NP ∩ co-NP.

The next Corollary gives our first upper bound on the size of memory sufficient for winning strategies in Muller games. Here we use the *Memoryless Determinacy Theorem* for Mostowski games due to Mostowski [Mos91] and Emerson and Jutla [EJ91]. Note that the Memoryless Determinacy Theorem, as well as Corollary 7 are special cases of Theorem 12 of the next section.

Corollary 7 Let $\mathcal{G} = (\mathcal{A}, \mathcal{F})$ be a Muller game with the winning condition $\mathcal{F}$. The game $\mathcal{G}$ is determined and both players have winning strategies with memory of size $l_{\mathcal{F}}$ from their winning sets.

Proof
Let $\mathcal{G}$ be a Muller game and let $\mathcal{G}'$ be the Mostowski game constructed form $\mathcal{G}$ as in the proof of Theorem 5. By the Memoryless Determinacy Theorem for Mostowski games both players have memoryless winning strategies from their winning sets in the game $\mathcal{G}'$. As observed in the proof of Theorem 5 a winning strategy for a player in the game $\mathcal{G}$ gives rise to a winning strategy for the same player in the game $\mathcal{G}'$, and vice versa. A winning strategy in the game $\mathcal{G}$ obtained in this way from a *memoryless* winning strategy in the game $\mathcal{G}'$ is in general *not memoryless*. It is, however, sufficient for this strategy to "simulate" the corresponding plays in $\mathcal{G}'$ by keeping track (in finite memory) of the second component of the vertices in $\mathcal{G}'$, i.e., the generalised LAR's. Hence the number $l_{\mathcal{F}}$ of generalised LAR's for the winning condition $\mathcal{F}$ is an upper bound on the size of memory sufficient for winning strategies in games with this winning condition. □

4 Upper bound

The proof of the upper bound on the size of memory for winning strategies in Muller games used to establish Corollary 7 is designed to work for both players at the same time. Whereas in Mostowski games both players have memoryless winning strategies, in Muller games in general there is an asymmetry between the players. For example if a winning condition of a player can be represented in Rabin form then by the results of Emerson [Eme85], Klarlund [Kla92] and McNaughton [McN93] the player has a memoryless winning strategy. The opponent of a player with a Rabin winning condition has a Street winning condition. Lescow [Les95] observed that there are Street winning conditions for which winning strategies require memory of size exponential in the number of colours. In

Section 5 we show a general lower bound which, in particular, implies a factorial lower bound for Street winning conditions.

In this section we address this asymmetry between the players. We reprove the *Forgetful Determinacy Theorem* of Gurevich and Harrington [GH82], while improving the upper bound on the size of memory obtained in Corollary 7. The improvement is due to a finer analysis of what data a player needs to keep track of in order to successfully play against the opponent. The crucial point is to treat the two players in a different manner, which gives way to considerable optimisations of the size of memory for winning strategies. Our upper bound is very much inspired by the work of Zielonka [Zie94]; in fact the proof of the Forgetful Determinacy Theorem we present here closely follows his. His elegant construction implicitly captures the asymmetry between the players we have mentioned above. We exploit the insight implicit in Zielonka's approach to infer the improved upper bound on the size of memory for winning strategies. In Section 5 we show that our upper bound is optimal.

We are going to determine for every winning condition $\mathcal{F}$ the number $m_{\mathcal{F}}$ such that for every game with the winning condition $\mathcal{F}$ the memory of size $m_{\mathcal{F}}$ is sufficient for a winning strategy in this game. The number $m_{\mathcal{F}}$ we are after can be obtained by looking at the structure of the Zielonka tree of the winning condition $\mathcal{F}$.

Definition 8 (The number $m_{\mathcal{F}}$) Let $\mathcal{F} \subseteq \mathcal{P}(C)$ be a winning condition and $\mathcal{Z}_{\mathcal{F}_0,C_0}, \ldots, \mathcal{Z}_{\mathcal{F}_{k-1},C_{k-1}}$ be the subtrees attached to the root of the tree $\mathcal{Z}_{\mathcal{F},C}$. By $\mathcal{F}_i$ we denote the condition $\mathcal{F} \restriction C_i \subseteq \mathcal{P}(C_i)$ for $i = 0, \ldots, k-1$. We define the number $m_{\mathcal{F}}$ as follows

$$m_{\mathcal{F}} = \begin{cases} 1 & \text{if } \mathcal{Z}_{\mathcal{F},C} \text{ does not have any subtrees,} \\ \max\{m_{\mathcal{F}_0}, \ldots, m_{\mathcal{F}_{k-1}}\} & \text{if } C \notin \mathcal{F}, \\ \sum_{i=0}^{k-1} m_{\mathcal{F}_i} & \text{if } C \in \mathcal{F}. \end{cases}$$

The following fact states explicitly that Theorem 12 provides an improvement over the upper bound of Corollary 7.

Fact 9 For every winning condition $\mathcal{F} \subseteq \mathcal{P}(C)$ we have $m_{\mathcal{F}} \leq l_{\mathcal{F}}$.

It can be proven for example by showing that $m_{\mathcal{F}}$ is the maximal number of leaves in a *1-subree* of the Zielonka tree $\mathcal{Z}_{\mathcal{F}}$, where by a 1-subtree we mean a subtree with every 1-level node having at most one child.

The proof of the main theorem proceeds by induction on the number of colours. In particular we will restrict a given arena to *subarenas* with smaller number of colours. A subgraph of an arena is a subarena if every node of the subgraph has a successor in the subgraph.

Before starting the main argument we need one more notion. Suppose we somehow know that a player can win from a set of vertices T in some subarena W. Clearly, if from some node of W this player can force a play into T then he also has a strategy from that vertex. The set of all such vertices will be called the *attractor* of T.

Definition 10 (Attractor) Let $\mathcal{A} = (V, V_0, V_1, E, C, \chi)$ be an arena. The *attractor for player i of a set T in the subarena $W \subseteq V$* is the set $Attr_i^W(T) = \bigcup_\xi T_\xi$ where the sequence $T_0, T_1, T_2, \ldots$ of sets of vertices is defnied by transfinite induction as follows:

- $T_0 = T \cap W$,
- $T_{\xi+1} = T_\xi \;\cup\; \{v \in V_i \cap W : \exists w.\, E(v,w) \wedge w \in T_\xi\}$
  $\qquad\qquad \cup\; \{v \in V_{1-i} \cap W : \forall w.\, E(v,w) \Rightarrow w \in T_\xi\}$,
- if ξ is a limit ordinal then let $T_\xi = \bigcup_{\rho<\xi} T_\rho$.

Lemma 11 For $\mathcal{A}$, W, T as above and $i \in \{0,1\}$. From every vertex $v \in Attr_i^W(T)$ player i has a memoryless strategy to bring the play to vertices in T. The set $W \setminus Attr_i^W(T)$ is an arena.

Theorem 12 (Forgetful Determinacy)
Let $\mathcal{G} = (\mathcal{A}, \mathcal{F})$ be a game with the arena $\mathcal{A} = (V, V_0, V_1, E, C, \chi)$ and the winning condition $\mathcal{F} \subseteq \mathcal{P}(C)$. Let $W_i \subseteq V$ for $i = 0, 1$ be the winning set of player i in the game $\mathcal{G}$. Then the following hold

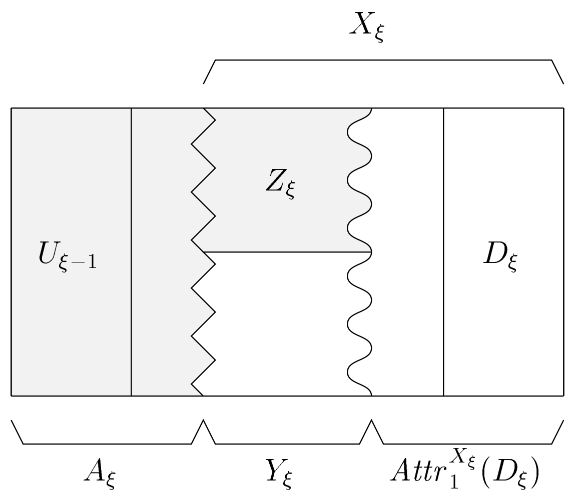
1. *$W_0 \cup W_1 = V$, i.e., the game $\mathcal{G}$ is* determined*,*
2. *player 0 has a winning strategy with memory of size $m_\mathcal{F}$ from the set W_0,*
3. *player 1 has a winning strategy with memory of size $m_{\overline{\mathcal{F}}}$ from the set W_1,*

where $\overline{\mathcal{F}} = \mathcal{P}(C) \setminus \mathcal{F}$.

Proof
The proof goes by induction on the structure of the Zielonka tree $\mathcal{Z}_{\mathcal{F},C}$ of the winning condition $\mathcal{F} \subseteq \mathcal{P}(C)$. We assume that $C \notin \mathcal{F}$. In case $C \in \mathcal{F}$ we exchange the roles of the players in the proof below, i.e., we consider the game $\overline{\mathcal{G}} = (\mathcal{A}, \overline{\mathcal{F}})$.

We define by transfinite induction the nondecreasing sequence of sets $\langle U_\xi \rangle$ as follows.

1. If ξ is a limit ordinal then let $U_\xi = \bigcup_{\nu<\xi} U_\nu$ (in particular, $U_0 = \emptyset$).
2. If ξ is a successor ordinal then we calculate U_ξ as follows (see Figure 3):
   (a) $A_\xi = Attr_0^V(U_{\xi-1})$,
   (b) $X_\xi = V \setminus A_\xi$,
   (c) $Y_\xi = X_\xi \setminus Attr_1^{X_\xi}\big(\chi^{-1}(C \setminus C_{(\xi \bmod k)})\big)$,
   (d) let Z_ξ be the winning set of player 0 in subgame $\mathcal{G}_\xi = (\mathcal{A} \restriction Y_\xi, \mathcal{F} \restriction C_{(\xi \bmod k)})$,
   (e) $U_\xi = A_\xi \cup Z_\xi$.



The shaded region is equal to U_ξ; $D_\xi = \chi^{-1}(C \setminus C_{(\xi \bmod k)})$.

Figure 3: Calculating U_ξ for ξ being a successor ordinal.

Observe that $\mathcal{G}_\xi$ is indeed a subgame, i.e., Y_ξ is a subarena. This is because X_ξ is a subarena as a complement of an attractor, and then Y_ξ is a subarena for the same reason.

By $\mathcal{F}_i$ we denote the winning condition $\mathcal{F} \restriction C_i \subseteq \mathcal{P}(C_i)$ for $i = 0, \ldots, k-1$. Let $\overline{\mathcal{F}_i} = \mathcal{P}(C_i) \setminus \mathcal{F}_i$ denote the complement of $\mathcal{F}_i$. Clearly, for every $i = 1, \ldots, k-1$ the Zielonka tree of the condition $\mathcal{F}_i$ is a subtree of the Zielonka tree for $\mathcal{F}$. For every ordinal ξ, the winning condition in the game $\mathcal{G}_\xi$ is $\mathcal{F}_i$, where $i = \xi \bmod k$. Hence by the induction hypothesis player 0 has a winning strategy with memory of the size $m_{\mathcal{F}_i}$ from the set Z_ξ, and player 1 has a winning strategy with memory of the size $m_{\overline{\mathcal{F}_i}}$ from the set $Y_\xi \setminus Z_\xi$.

Define $W = \bigcup_\xi U_\xi$. We now show (Fact 13) that player 0 has a winning strategy from W in the game $\mathcal{G}$ with memory of size $m_\mathcal{F}$. Later on we complement this by showing (Fact 14) that player 1 has a winning strategy from $V \setminus W$ in the game $\mathcal{G}$ with memory of size $m_{\overline{\mathcal{F}}}$. This establishes determinacy of the game $\mathcal{G}$ with the winning sets of the players being $W_0 = W$ and $W_1 = V \setminus W$, and settles the expected upper bounds on the size of memory for winning strategies.

Fact 13 Player 0 has a winning strategy from W_0 using memory of size $m_\mathcal{F}$.

Proof: Let $M_\mathcal{F} = \{1, \ldots, m_\mathcal{F}\}$. We show by transfinite induction that for every ordinal ξ, player 0 has a winning strategy $\theta^\xi = (\theta^\xi_M, \theta^\xi_V)$ from the set U_ξ in the game $\mathcal{G}$, such that if player 0 uses this strategy, then the play stays forever inside U_ξ (player 1 cannot force the play outside U_ξ because U_ξ is a trap for him). Moreover this strategy uses only memory $M_\mathcal{F}$ and does not depend on the initial memory (see remark on page 3). The sequence $\langle U_\xi \rangle$ is nondecreasing, hence there is an ordinal ρ with $U_\rho = W$ and we can take the strategy $(\theta^\rho_M, \theta^\rho_V)$ as the winning strategy for player 0 from W.

1. Suppose ξ is a limit ordinal. For every vertex $v \in U_\xi$ we define $\mu(v)$ to be the smallest ordinal ρ for which $v \in U_\rho$. We define the strategy with memory $(\theta^\xi_M, \theta^\xi_V)$ by setting for every $v \in U_\xi$:

   $$\theta^\xi_M(m, v) = \theta^{\mu(v)}_M(m, v) \quad \text{and} \quad \theta^\xi_V(v, m) = \theta^{\mu(v)}_V(v, m)$$

   To see that this is a winning strategy for player 0, take a play $\pi = \langle v_0, v_1, v_2, \ldots \rangle$ consistent with the strategy. Observe that the sequence $\mu(v_0), \mu(v_1), \mu(v_2), \ldots$ is nonincreasing, because for every $\rho < \xi$ the strategy θ^ρ forces the play to stay inside U_ρ. Ordinals are well founded, so the sequence $\mu(v_0), \mu(v_1), \mu(v_2), \ldots$ is eventually constant. Hence, the play π is eventually consistent with a strategy θ^ρ for some $\rho < \xi$, and player 0 wins the play π.

2. Suppose ξ is a successor ordinal. By the induction hypothesis player 0 has a winning strategy $\theta^{\xi-1}$ from the set $U_{\xi-1}$ in the game $\mathcal{G}$. From the set $A_\xi = Attr^V_0(U_{\xi-1})$ player 0 has a winning strategy consisting of bringing the play to $U_{\xi-1}$ and then using $\theta^{\xi-1}$. Let κ_ξ be the memoryless strategy "attract to the set $U_{\xi-1}$" defined on the set $A_\xi \setminus U_{\xi-1}$. Denote by λ_ξ the winning strategy for player 0 from the set Z_ξ in the subgame $\mathcal{G}_\xi$.

   We now show that taken together, the strategies $\theta^{\xi-1}$, κ_ξ, and λ_ξ give a a winning strategy for player 0 from the set $U_\xi = A_\xi \cup Z_\xi$ in the game $\mathcal{G}$. If the play starts in a vertex from Z_ξ then player 0 uses λ_ξ. If the play stays forever in the set Z_ξ then it is consistent with the strategy λ_ξ which is winning for player 0. If the play leaves Z_ξ then we it must be the case that player 1 chooses a vertex in A_ξ. Player 1 cannot move to $Y_\xi \setminus Z_\xi$, because Z_ξ as the winning set of player 0 in game $\mathcal{G}_\xi$ is a trap for player 1. A move to $X_\xi \setminus Y_\xi$ is not possible either, because Y_ξ is a trap for player 1 as a complement of an attractor for player 0. But in A_ξ player 0 may use her memoryless strategy κ_ξ to bring the game to $U_{\xi-1}$, and then use her winning strategy $\theta^{\xi-1}$ and thus win.

   Let us define θ^ξ formally. The strategy $\lambda_\xi = (l^\xi_M, l^\xi_V)$ is obtained from the induction hypothesis; it is defined on Z_ξ and uses memory of size $m_{\mathcal{F}_i}$, where $i = \xi \bmod k$. We may for example assume that the memory $M_{\mathcal{F}_i}$ of λ_ξ is the set $\{1, \ldots, m_{\mathcal{F}_i}\}$. The strategy κ_ξ is memoryless so it is a function $\kappa_\xi : A_\xi \setminus U_{\xi-1}) \to A_\xi$. We set

$$\theta^\xi_M(m,v) = \begin{cases} \theta^{\xi-1}_M(m,v) & \text{if } v \in U_{\xi-1}, \\ l^\xi_M(m,v) & \text{if } v \in Z_\xi \text{ and } m \in M_{\mathcal{F}_i}, \\ 1 & \text{otherwise,} \end{cases}$$

$$\theta^\xi_V(v,m) = \begin{cases} \theta^{\xi-1}_V(v,m) & \text{if } v \in U_{\xi-1}, \\ l^\xi_V(v,m) & \text{if } v \in Z_\xi \text{ and } m \in M_{\mathcal{F}_i}, \\ l^\xi_V(v,1) & \text{if } v \in Z_\xi \text{ and } m \notin M_{\mathcal{F}_i}, \\ \kappa_\xi(v) & \text{otherwise.} \end{cases}$$

It can be checked that the strategy θ^ξ defined in this way corresponds to the informal description we have given above.

□

Fact 14 Player 1 has a winning strategy on $V \setminus W$ using memory of size $m_{\overline{\mathcal{F}}}$.

Proof: Let μ be an ordinal such that $\mu \bmod k = 0$ and $W = U_{\mu-1} = U_\mu = \cdots = U_{\mu+k-1}$. From the equality $W = U_{\mu-1} = U_\mu$ we have that $Attr^V_0(W) = W$. Hence $V \setminus W$ is a subarena of $\mathcal{G}$ and a trap for player 0. The equality $U_{\mu+i-1} = U_{\mu+i}$ implies that $Z_{\mu+i} = \emptyset$. So from the induction hypothesis it follows that for every $i = 0, \ldots, k-1$ player 1 has a winning strategy θ^i with memory of size $m_{\overline{\mathcal{F}_i}}$ from the set $Y_{\mu+i}$ in the game $\mathcal{G}_{\mu+i}$.

The strategy for player 1 on $V \setminus W$ is the following. If a play starts in a vertex of some of the sets $Y_{\mu+i}$ for $i = 0, 1, \ldots, k-1$, then he should use the strategy θ^i. If the play stays forever in $Y_{\mu+i}$ then player 1 wins, because θ^i is a winning strategy for him from $Y_{\mu+i}$ in $\mathcal{G}_\xi$. The play may leave $Y_{\mu+i}$ only because player 0 chooses a vertex outside. This vertex cannot belong to W, because $V \setminus W$ is a trap for player 0. It cannot be in $Y_{\mu+i} \setminus Z_{\mu+i}$ either, because $Z_{\mu+i}$ as the winning set of player 1 in $\mathcal{G}_{\mu+i}$ is a trap for player 0. Hence it must be in $Attr^{V\setminus W}_1\big(\chi^{-1}(C \setminus C_i)\big)$. From there player 1 has a strategy κ_i to reach a vertex in $\chi^{-1}(C \setminus C_i)$, without leaving $V \setminus W$. Such a vertex belongs to $Y_{\mu+(i+1) \bmod k}$ and player 1 can use the strategy $\theta^{(i+1) \bmod k}$ from this place. Of course, after some moves, the play may leave $Y_{\mu+(i+1) \bmod k}$ and we would repeat the above reasoning to force the play into $Y_{\mu+(i+2) \bmod k}$, and so on.

If the play finally settles in some $Y_{\mu+i}$ then from this moment it is consistent with θ_i and player 1 wins. If the play infinitely often jumps between subsequent $Y_{\mu+i}$ then for every $i = 0, \ldots, k-1$ the play infinitely often visits a node with a colour not in C_i. Hence the set of frequent colours of the play is not contained in C_i. As $C_0, \ldots, C_k$ are all maximal subsets of $\mathcal{F}$ we have that the set of frequent colours of the play is not in $\mathcal{F}$ and the play is winning for player 1.

To formally define the strategy described above we must be more precise with the descriptions of the strategies θ^i. The strategy $\theta^i = (\theta^i_M, \theta^i_V)$ uses the memory $M_{\overline{\mathcal{F}}_i} = \{1, \ldots, m_{\overline{\mathcal{F}}_i}\}$. By the definition of $m_{\overline{\mathcal{F}}}$ we have $m_{\overline{\mathcal{F}}} = \sum_{i=0}^{k-1} m_{\overline{\mathcal{F}}_i}$. Let

$M_{\overline{\mathcal{F}}} = \bigcup_{i=0}^{k-1}(M_{\overline{\mathcal{F}}_i} \times \{i\})$. This is not exactly the set $\{1, \dots, m_{\overline{\mathcal{F}}}\}$ but it has the same cardinality which is enough for out purposes.

We define the strategy $\theta = (\theta_M, \theta_V)$ as follows:

$$\theta_M^i(v,(m,i)) = \begin{cases} \theta_M^i(v,(m,i)) & v \in Y_{\mu+i} \\ (1, i+1 \bmod k) & \text{otherwise} \end{cases}$$

$$\theta_V(v,(m,i)) = \begin{cases} \theta_V^i(v,(m,i)) & v \in Y_{\mu+i} \\ \kappa(v) & v \in Attr_1^{W_1}(T_i) \setminus T_i \\ w_i & v \in T_i \end{cases}$$

where $w_i \in W_1$ is an arbitrary chosen successor of v. Such a successor exists because W_1 is an arena. It can be checked that the strategy θ^ξ defined by these equations corresponds the informal description we have given above. Please observe that we needed to take a disjoint sum of all $M_{\overline{\mathcal{F}}_i}$ only because a $Y_\mu, \dots, Y_{\mu+k-1}$ may be not disjoint and in our strategy we needed to know on which Y_j we are playing. □ □

Remark: Note that, in particular, if $\mathcal{P}(C) \setminus \mathcal{F}$ is closed under union, i.e., $\mathcal{F}$ can be expressed by a *Rabin condition* (see [Zie94]), then the strategy $\zeta_{\mathcal{G}}$ is memoryless. Memoryless determinacy for such games was shown in [Eme85, Kla92].

Remark: Observe that every 1-subtree of a tree from the Example on page 4 (Figure 1) has n leaves (labelled with $H_{i1}, \dots, H_{in}$ for some i). From Theorem 12 it follows that winning strategies for player 0 in every game with such a winning condition need at most memory of size $\mathcal{O}(n)$.

By a careful analysis of the strategy synthesis algorithm implicit in the definition of the strategy $\zeta_{\mathcal{G}}$ one can obtain the following fact.

Corollary 15 (Strategy synthesis algorithm) There is an algorithm that given a game $\mathcal{G} = (\mathcal{A}, \mathcal{F})$ computes a winning strategy for player 0 in time $\mathcal{O}((|\mathcal{A}| * d)^h)$; here d is the maximum degree of a node and h is the height of the Zielonka tree $\mathcal{Z}_{\mathcal{F}}$.

5 Lower bounds

5.1 The $m_{\mathcal{F}}$ lower bound

Here we consider the lower bound on the size of memory needed by a winning strategy in case when the size of a graph is not limited by the number of colours. We show that in this case the upper bound of Theorem 12 is tight.

Theorem 16 (Lower bound)
For every winning condition $\mathcal{F}$ there is an arena $\mathcal{A}_{\mathcal{F}}$ such that every winning strategy for player 0 in the game $(\mathcal{A}_{\mathcal{F}}, \mathcal{F})$ requires memory of size at least $m_{\mathcal{F}}$.

Proof
Let $\mathcal{F} \subseteq \mathcal{P}(C)$. Assume that $C \in \mathcal{F}$. Otherwise we take a son s' of the root, such that, there is a 1-subtree with $m_{\mathcal{F}}$ leaves below s'. Then we can take the label C' of s' and continue our construction for $\mathcal{F} \upharpoonright C'$ instead of $\mathcal{F}$.

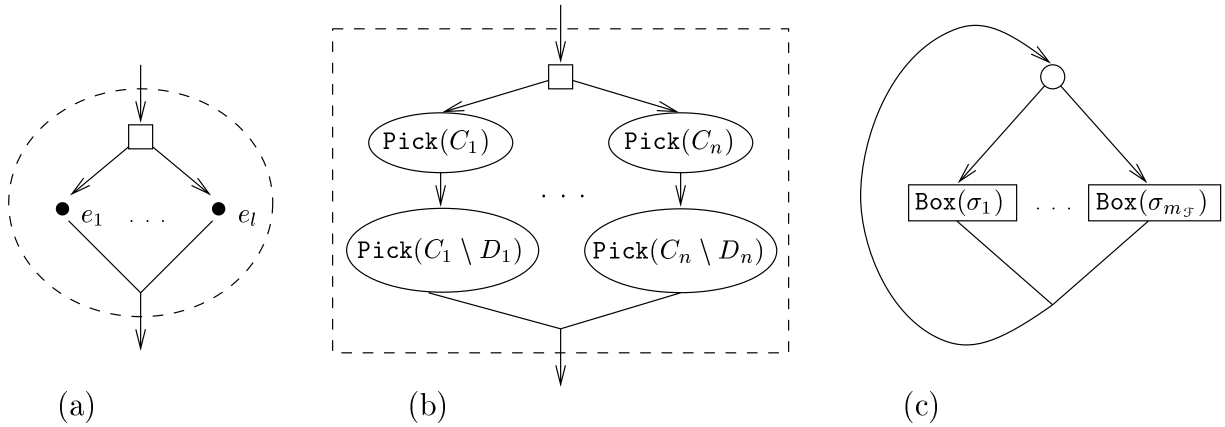


Figure 4: The arena $\mathcal{A}_{\mathcal{F}}$.

Let T be a 1-subtree of the Zielonka tree $\mathcal{Z}_{\mathcal{F}}$ with $m_{\mathcal{F}}$ leaves and let $\sigma_1, \ldots, \sigma_{m_{\mathcal{F}}}$ be all the leaves of T. We define the arena $\mathcal{A}_{\mathcal{F}}$ as follows. For every set of colours $E = \{e_1, \ldots, e_l\}$ we define a subarena $\mathtt{Pick}(E)$ as on Figure 4(a). In the square (uncoloured) vertex player 1 makes a choice to visit a vertex coloured with one of the colours in E. To facilitate the description of the whole arena let us use the sequence of labels on the path leading to a node of T to identify this node. A notation $C_1D_1 \ldots C_nD_n(C_{n+1})$ denotes a leaf to which leads a path with labels $C_1D_1 \ldots C_nD_n$ or $C_1D_1 \ldots C_nD_nC_{n+1}$. For every leaf $\sigma = C_1D_1 \ldots C_nD_n(C_{n+1})$ of T we define a subarena $\mathtt{Box}(\sigma)$ as on Figure 4(b). Again, in the square (uncoloured) vertex player 1 makes a move. Finally the arena $\mathcal{A}_{\mathcal{F}}$ is defined as on Figure 4(c), where in the circle (uncoloured) vertex player 0 makes a move.

Observe that a play on the arena $\mathcal{A}_{\mathcal{F}}$ consists of an infinite sequence of the following choices:

1. Player 0 chooses $\mathtt{Box}(\sigma)$ for some leaf $\sigma = C_1D_1 \ldots C_nD_n(C_{n+1})$ of T.
2. Player 1 chooses one of the indices $j \in \{1, \ldots, n\}$.
3. Player 1 picks a vertex coloured with one of the colours in C_j and then a vertex coloured with one of the colours in $C_j \setminus D_j$.

Claim 16.1 Player 0 has a winning strategy in the game $(\mathcal{A}_{\mathcal{F}}, \mathcal{F})$.

Proof: The only significant choices of player 0 are made in step 1. In the strategy we are defining her choice will depend on the choice made by player 1 in step 2 of the preceding round of the play. Suppose that player 0 has chosen $\texttt{Box}(\sigma)$ and player 1 has chosen the index j in step 2. Then in the next round player 0 chooses $\texttt{Box}(\sigma')$, where σ' is some leaf in the subtree of T rooted in the next child of the node $C_1D_1 \dots C_j$, after the child $C_1D_1 \dots C_jD_j$. (If $C_1D_1 \dots C_jD_j$ is the last child of its parent then pick a leaf σ in the subtree rooted in the first child of $C_1D_1 \dots C_j$.)

Now we will argue that the strategy just described is winning for player 0. Consider a play π consistent with the strategy. Let $\gamma = C_1D_1 \dots C_i$ be the root of the lowest subtree of T containing all the leaves of T chosen infinitely often by player 0 in step 1. Let $\gamma_j = C_1D_1 \dots C_iD_{i_j}$ for $j = 1, \dots, k$ be all the children of γ in T. Observe that for every $j = 1, \dots, k$ there is a leaf σ_j in the subtree rooted in γ_j, such that the index i is chosen by player 1 in step 2 infinitely often in $\texttt{Box}(\sigma_j)$. This, however, implies that $\mathit{Inf}(\pi) \not\subseteq D_{i_j}$ for every $j = 1, \dots, k$. On the other hand it is not hard to see that $\mathit{Inf}(\pi) \subseteq C_i$, hence $\mathit{Inf}(\pi) \in \mathcal{F}$. $\square$

If the size of the memory of a strategy for player 0 in the game $(\mathcal{A}_{\mathcal{F}}, \mathcal{F})$ is less than $m_{\mathcal{F}}$, then clearly there is a leaf σ of T, such that player 0 playing according to this strategy never visits $\texttt{Box}(\sigma)$. Hence, the following claim establishes the lower bound of the theorem.

Claim 16.2 Consider a strategy for player 0 in the game $(\mathcal{A}_{\mathcal{F}}, \mathcal{F})$ and a leaf σ in the tree T. If player 0 never visits $\texttt{Box}(\sigma)$ when playing according to the strategy, then the strategy is not winning for player 0.

Proof: Let $\sigma = C_1D_1 \dots C_nD_n(C_{n+1})$. We will now provide a counter-strategy for player 1 allowing him to make one of D_i's the set of frequent colours. Which D_i it will turn out to be depends on the actual behaviour of the strategy of player 0.

To play according to his counter-strategy player 1 keeps in his memory a tuple $\vec{d} \in D_1 \times \dots \times D_n$ to remember which element of each D_i is currently his target.

Assume that player 0 chooses $\texttt{Box}(\delta)$ such that $\delta = C_1D_1 \dots C_jD'_j \dots$ with $D'_j \neq D_j$, i.e., j is the smallest index at which σ and δ differ. Player 1 responds making the following choices in $\texttt{Box}(\delta)$:

1. Choose $\texttt{Pick}(C_j)$ and then pick a vertex coloured with d_j, where $(d_1, \dots, d_n)$ is the current memory of player 1.

2. From $\texttt{Pick}(C_j \setminus D'_j)$ choose a vertex coloured with some $c \in D_j$. It is possible to do so, because $D_j \subset C_j$ and $D_j \setminus D'_j \neq \emptyset$.

After performing these moves player 1 updates his current memory $(d_1, \dots, d_n)$ by changing its j-th component to the next element of D_j in some fixed linear order (or to the minimal element of D_j in this order, if d_j was maximal).

Now we will argue that this counter-strategy guarantees a win for player 1. For a play π formed according to this counter-strategy consider the set I of indices j chosen infinitely often in the way described above. Set $i = \min(I)$ and observe that from some moment of the play π only colours in the set $D_i \cup D_{i+1} \cup \cdots \cup D_n = D_i$ appear (recall that $D_l \supset D_{l+1}$ for $l = 1, \ldots, n-1$). Moreover, each colour of D_i is hit infinitely often. Thus $Inf(\pi) = D_i$. This completes the proof since D_i is a winning set of player 1. □ □

Remark: The arena $\mathcal{A}_\mathcal{F}$ constructed in the proof of Theorem 16 is roughly of size $m_\mathcal{F}$. Hence, if for a family of conditions $\langle \mathcal{F}_n \rangle_{n \in \mathbb{N}}$ the number $m_{\mathcal{F}_n}$ grows superpolynomially in the number of colours of $\mathcal{F}_n$, then our lower bound does not apply to the case when we restrict ourselves to arenas of size polynomial in the number of colours. We do not know whether the $m_\mathcal{F}$ bound is strict in this case.

5.2 Optimality of the LAR's

Now we show a factorial lower bound establishing optimality of the LAR data structure even for games with arena sizes bounded linearly in the number of colours of the winning condition. Optimality of LAR's in the case when we do not bound the size of the game follows from the previous section.

Theorem 17 (LAR's are optimal)
There is a family of games $\langle \mathcal{G}_n \rangle_{n \in \mathbb{N}}$, such that the arena of $\mathcal{G}_n$ is of size $\mathcal{O}(n)$ and every winning strategy for player 0 in $\mathcal{G}_n$ has memory of size at least $n!$.

Proof
The game $\mathcal{G}_n$ has the arena $\mathcal{A}_n = (V, V_0, V_1, E, C_n, \chi)$, where: $V_0 = \{-n, \ldots, -1\}$, $V_1 = \{1, \ldots, n\}$, $E = \big\{(-i, j), (i, -j) : i, j \in \{1, \ldots, n\}\big\}$ and $C_n = V$. We identify vertices with colours setting $\chi(v) = v$, for every $v \in V$. The winning condition of $\mathcal{G}_n$ is defined as $\mathcal{F}_n = \{C \subseteq C_n : |C \cap V_0| = \max(C \cap V_1)\}$.

It is not difficult to see that player 0 has an LAR winning strategy in $\mathcal{G}_n$ (in fact it suffices to consider LAR's over colours from the set V_0 only).

Suppose that $\mathcal{S} = (M, V_0, V_1, m_I, \Rightarrow)$ is an I/O automaton realizing a strategy for player 0. For the sake of this proof by the *transition graph of* $\mathcal{S}$ we will understand the graph $(M, \rightarrow)$ with edges labelled by elements of V_0, where $m \stackrel{v_0}{\rightarrow} m'$ if there is a $v_1 \in V_1$, such that $m \underset{v_1}{\stackrel{v_0}{\Rightarrow}} m'$ holds.

In the following lemma we slightly strengthen the statement of the theorem, in order for the proof by induction on n to go through.

Lemma 18 Let $\mathcal{S}$ be the I/O automaton realizing a winning strategy for player 0 in $\mathcal{G}_n$. Then there is a state $m \in M$ reachable from the initial state of the transition graph of $\mathcal{S}$, and a finite sequence $w \in V_0^*$ with at least one occurrence of every element of V_0, such that:

1. $m \stackrel{w}{\rightarrow}_* m$, i.e., the path in the transition graph of the automaton $\mathcal{S}$, labelled with the input word w and starting in state m is a loop,
2. $|M_w| \geq n!$, where M_w is the set of states of $\mathcal{S}$ on the path described above.

Proof: W.l.o.g. we can assume that the transition graph of the automaton $\mathcal{S}$ is strongly connected. Otherwise we could restrict the transition graph of $\mathcal{S}$ to one of its strongly connected components (s.c.c.), that is reachable from the initial state, and is final in the acyclic graph of all s.c.c.'s of $\mathcal{S}$ (i.e., there are no edges going out of this s.c.c. in the transition graph of $\mathcal{S}$). As player 1 can in a finite number of moves force the play into a reachable final s.c.c., clearly after such a restriction the remaining strategy is still winning for player 0.

The base case of the induction follows immediately from the assumption that the transition graph of $\mathcal{S}$ is strongly connected.

Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be sub-games of $\mathcal{G}_n$, such that $\mathcal{H}_i$ is obtained by removing $-i$ from V_0 and n from V_1. It is easy to see that I/O automata $\mathcal{S}_1, \ldots, \mathcal{S}_n$ obtained by straightforward restrictions[1] of $\mathcal{S}$ realize winning strategies for player 0 in games $\mathcal{H}_1, \ldots, \mathcal{H}_n$. From the induction hypothesis it follows that there are states $m_1, \ldots, m_n \in M$, and finite sequences $w_1, \ldots, w_n$, such that $w_i \in (V_0 \setminus \{-i\})^*$ and $|M_i| \geq (n-1)!$, where M_i is the set of states on the path $m_i \stackrel{w_i}{\to}_* m_i$.

Observe that the outputs produced by $\mathcal{S}$ on any path $m_i \stackrel{w_i}{\to}_* m_i$ for $i = 1, \ldots, n$ do not contain any occurrence of $n \in V_1$. Otherwise, the infinite input sequence $(w_i)^\omega$ would form a play winning for player 1. This, however, would contradict the assumption that $\mathcal{S}$ realized a winning strategy for player 0 in $\mathcal{G}_n$.

We will now argue, that if $i \neq j$ then $M_i \cap M_j = \emptyset$. This would immediately imply that $\left|\bigcup_{i=1}^n M_i\right| \geq n \cdot (n-1)! = n!$. Assume the contrary, i.e., that there is a state $m \in M_i \cap M_j$ for some $i \neq j$. Then there are partitions $w_i = u_1 \cdot u_2$ and $w_j = v_1 \cdot v_2$, such that $m_i \stackrel{u_1}{\to}_* m \stackrel{u_2}{\to}_* m_i$ and $m_j \stackrel{v_1}{\to}_* \ m \stackrel{v_2}{\to}_* m_j$. So there is a loop $m \stackrel{u_2 \cdot u_1 \cdot v_2 \cdot v_1}{\longrightarrow_*} m$. Hence, the infinite input sequence $(u_2 \cdot u_1 \cdot v_2 \cdot v_1)^\omega$ induces a play winning for player 1, because the sequence $u_2 \cdot u_1 \cdot v_2 \cdot v_1$ contains occurrences of all n elements of V_0, and the output generated by $\mathcal{S}$ does not contain a occurrence of $n \in V_1$. This, however, contradicts the assumption that $\mathcal{S}$ realized a winning strategy for player 0.

To finish the proof of the Lemma it remains to show that the sequences w_i for $i = 1, \ldots, n$ can be composed to yield a sequence w, such that $m \stackrel{w}{\to}_* m$ for some $m \in M$ and this cycle includes all cycles $m_i \stackrel{w_i}{\to}_* m_i$ as its sub-paths. Again, the assumption of strong connectedness of the transition graph of $\mathcal{S}$ makes this task trivial. □ □

6 Acknowledgements

The first two authors are grateful to Damian Niwiński for spiritual support and a wonderful introduction to the theory of automata on infinite objects.

[1] More precisely, in order to obtain $\mathcal{S}_i$ we restrict the input alphabet of $\mathcal{S}$ to $V_0 \setminus \{-i\}$, and change the output component of the transition relation whenever it is n to, for example, $(n-1)$.

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