Concurrent Parity Games

Reducing Concurrent to Turn-Based Büchi and Co-Büchi Games

Marcin Jurdziński  Orna Kupferman  Tom Henzinger

LIAFA, Paris 7  UC Berkeley
Parity Games

Focus here on:

- generalization to concurrent games
  (probabilistic strategies needed)
- generalization of determinacy
- structure of winning strategies
- algorithmics of solving games
**Parity Game Example**

- **Play:** Infinite path
- **Winning Play (for Pl.0):**
  - Highest priority occurring infinitely often is even
- **Strategy**
- **Winning Strategy**
### Strategies

<table>
<thead>
<tr>
<th></th>
<th>History</th>
<th>Current Vertex</th>
<th>Next Vertex</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Pure</strong></td>
<td></td>
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<tr>
<td>General</td>
<td>$V^*$</td>
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<td>w/memory</td>
<td>$M$</td>
<td>$V$</td>
<td>$V \times M$</td>
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<tr>
<td>Memoryless</td>
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<td>$V$</td>
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<tr>
<td><strong>Outcome</strong></td>
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<td></td>
<td></td>
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<tr>
<td>$(v, \sigma, \tau)$: An Infinite Path</td>
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<tr>
<td><strong>Mixed</strong></td>
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<td></td>
<td></td>
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<tr>
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<td>$V^*$</td>
<td>$V$</td>
<td>$\text{Distr}(V)$</td>
</tr>
<tr>
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<td>$M$</td>
<td>$V$</td>
<td>$\text{Distr}(V) \times 1$</td>
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<tr>
<td><strong>Outcomes</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(v, \sigma, \tau)$: Probability Space Over Infinite Paths</td>
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Determinacy of turn-based parity games

\text{THM [EJ'91, Mos'91]}

There is a unique partition

\[ V = W_0 \cup W_1 \]

s.t. player \( i \) wins on \( W_i \)

Winning strategies for both players are

- \text{Pure}
- \text{Memoryless}
Concurrent Parity Games [dAMK'98c]

**Fact** For each pure strategy $\alpha$ for player 0, there is a strategy $\beta$ for player 1, s.t. $\text{Outcome}(\omega, \alpha, \beta)$ is losing for pl. 0

$\Rightarrow$ Pure strategies determinacy fails for concurrent games
A probabilistic strategy for PL. 0

Mixed strategy \( \alpha \) for player 0:

\[
\begin{align*}
\mathbb{P}^{\alpha}[\text{throw L}] &= \varepsilon \\
\mathbb{P}^{\alpha}[\text{throw R}] &= 1 - \varepsilon
\end{align*}
\]

Fact: For every mixed strategy \( \tau \) of PL. 1

\[
\mathbb{P}^{\alpha, \tau}[\text{hit}] \geq \varepsilon
\]

if \( \varepsilon \leq \frac{1}{2} \)

Corollary:

\[
\mathbb{P}^{\alpha, \tau}[\text{PL. 0 wins}] = 1
\]
Almost sure / positive probability win
"determinacy" for concurrent parity games

**Thm [dAH’00]** There is a unique partition

\[ V = W_0^S \cup W_1^p \]  

s.t.

- **Player 0 has a strategy** \( \alpha \) on \( W_0^S \), s.t.
  \[ P^{\alpha,\tau}_{1} [\text{pl. 0 wins}] = 1 \]
  for every strategy \( \tau \) for Player 1

- **Player 1 has a strategy** \( \beta \) on \( W_1^p \), s.t.
  \[ P^{\sigma,\beta}_{1} [\text{pl. 1 wins}] > 0 \]
  for every strategy \( \sigma \) for Player 0

Winning strategies \( \alpha, \beta \) are:

- Mixed
- With "counting" memory (infinite!)
"Almost" almost sure win

Player 0:
- Run, Stay

Player 1:
- Throw, Keep

Fact 1: Player 0 cannot win almost surely.

Fact 2: For every \( \epsilon > 0 \), player 0 has a strategy to win with probability \( \geq 1 - \epsilon \)

(limit sure win)
**Concurrent Reachability Game: Limit-Sure Win**

\[ \alpha: \begin{cases} \text{RUN} &\rightarrow& \varepsilon \\ \text{WAIT} &\rightarrow& 1-\varepsilon \end{cases} \quad \beta: \begin{cases} \text{THROW} &\rightarrow& 1-\kappa \\ \text{KEEP} &\rightarrow& \kappa \end{cases} \]

**Fact**

If player 0 plays \( \alpha \) in all steps, then he wins with probability \( \geq 1-2\varepsilon \).

**Claim**

1. \( P^{\alpha \beta}[\text{WIN}] \geq \varepsilon \)
2. \( P^{\alpha \beta}[\text{LOSE}] \leq 2\varepsilon \cdot P^{\alpha \beta}[\text{WIN}] \)

**Proof**

\( P^{\alpha \beta}[\text{WIN}] = \varepsilon \cdot \kappa + (1-\varepsilon) \cdot (1-\kappa) \geq \varepsilon \)

\( \frac{P^{\alpha \beta}[\text{LOSE}]}{P^{\alpha \beta}[\text{WIN}]} \leq \frac{\varepsilon \cdot (1-\kappa)}{(1-\varepsilon) \cdot (1-\kappa)} \leq 2\varepsilon \)

Ass. \( \varepsilon \leq \frac{1}{2} \)
**Concurrent Reachability Game: Limit-Sure Win**

**Diagram:**
- **Start State:** Lose
- **Actions:** Run, Throw, Wait, Keep
- **Transitions:**
  - Run, Throw → Lose
  - Run, Keep → Lose
  - Wait, Throw → Win

**Transition Matrix:**
\[ \alpha = \begin{bmatrix} \text{Run} & \epsilon \\ \text{Wait} & 1-\epsilon \end{bmatrix} \]

**Fact:** If Player 0 plays \( \alpha \) in all steps, then he wins with probability \( \geq 1-2\epsilon \).

**Claim:**
1. \( P^{x_0} [\text{win}] \geq \epsilon \)
2. \( P^{x_0} [\text{lose}] \leq 2\epsilon \cdot P^{x_0} [\text{win}] \)

**Proof:**
- \( L \) is the probability of reaching Lose before Win:
  \[ L = P^{x_0} [\text{lose}] + (1 - P^{x_0} [\text{win}]) \cdot L \]

\[ \Rightarrow L \leq \frac{P^{x_0} [\text{lose}]}{P^{x_0} [\text{win}]} \leq 2\epsilon \]
Concurrent Büchi game: Limit-Sure Win

Infinite Memory Strategy

\[ \alpha_\delta : \text{str. for pl. } O \text{ to run home with prob. } > \delta \]

\[ \alpha : \text{after running home } k \text{ times play } \alpha_\delta \]

Where \[ \delta_k \overset{df}{=} (1-\varepsilon)^{1/2^{k+1}} \]

Fact: Player 0 using \( \alpha \) runs home \( \infty \) many times with probability \( < 1-\varepsilon \)

Proof: \( R_k : \text{event that pl. } O \text{ runs home } \geq k \text{ times} \)

\[ P^{\alpha_k}[R_\infty] = \prod_{i=0}^{\infty} P^{\alpha}[R_{i+1} \mid R_i] \geq \prod_{i=0}^{\infty} \delta_i = 1-\varepsilon \]
**Limit sure/positive bounded probability win**

"Determinacy" for concurrent parity games

**Thm [DAH'00]** There is a unique partition

\[ V = W_0^{ls} \cup W_1^b \]

s.t.

- Player 0 wins limit-surely on \( W_0^{ls} \), i.e.

  For all \( \varepsilon > 0 \), he has a strategy \( \alpha \) on \( W_0^{ls} \), s.t.

  \[ \inf_{\tau} P^\tau_{x,\tau}[\text{pl. 0 wins}] > 1 - \varepsilon \]

- Player 1 wins with prob. bounded away from 0 on \( W_1^b \), i.e.

  There is \( k > 0 \), s.t. he has a strat. \( \beta \) on \( W_1^b \), s.t.

  \[ \inf_{\tau} P_0^\tau,\beta[\text{pl. 1 wins}] \geq k \]
"Quantitative" determinacy for concurrent probabilistic parity games

**THM [dAM’01]**

For each vertex $v$, there is a number $\pi_v$, s.t.

1. $\sup_{\tau} \inf_{g} P_{v}^{\sigma,\tau}[\text{pl. 0 wins}] = \pi_v$
2. $\sup_{\tau} \inf_{g} P_{v}^{\sigma,\tau}[\text{pl. 1 wins}] = 1 - \pi_v$

**Open Problem**

Efficient algorithms for solving quantitative concurrent parity games
The Main Result

Linear time reductions

- From concurrent Büchi games to turn-based Büchi games
- From concurrent co-Büchi games to turn-based Parity (0,2) games

Corollary

Concurrent co-Büchi games can be solved in quadratic time
Consequences and by-products

- The reductions turn solvers of turn-based parity games into solvers of concurrent Büchi and co-Büchi games

- A generalization of local witnesses (a.k.a. progress measures/signatures) to concurrent games

- An alternative proof of almost sure/positive prob. win duality from memoryless determinacy of turn-based parity games
Towards the translation

Concurrent Büchi Game

Player i wins from \( \mathcal{G} \)

Turn-based Büchi Game

Player i wins from \( \overline{\mathcal{G}} \)

Memoryless determinacy through local witnesses

Generalized local witnesses

Local witnesses

\( (\overline{\varphi}, \overline{\psi}) \)
The ingredients we need

1. Definition of local witnesses for concurrent games

2. Correctness proof for local witnesses: they induce winning strategies

3. Proof that local witnesses in $G$ induce local witnesses in $G$
Local Witnesses (for Büchi Games)

Player 0

\[ \phi : V \rightarrow \mathbb{N} \cup \{\infty\} \]

Player 1

\[ \psi : V \rightarrow \mathbb{N} \cup \{\infty\} \]
GENERALIZED LOCAL WITNESSES
FOR PLAYER 0 IN $C_{0,1}$ GAMES

$\varphi : V \rightarrow \mathbb{N} \cup \{\infty\}$

FACT PLAYER 0 WINS WITH PROBABILITY 1

PROOF

$R_k$: EVENT THAT RANK 0 IS REACHED $\geq k$ TIMES

$P[R_\infty] = \prod_{i=0}^{\infty} P[R_{i+1}, IR_i] = 1$
Generalized Local Witnesses
for Player 1 in $C_{as}(0,1)$ Games

$\Psi : \nu \rightarrow \mathbb{N} \cup \{\infty\}$

Fact: Player 1 wins with probability $> 0$

Proof (Induction on Rank)

- Rank $n = 0$

Claim: Player 1 can stay in $\Psi_0$ with prob. $\geq 1 - \varepsilon$

Proof

$\bar{\alpha}$: in k-th step choose $\alpha \varepsilon_k$

where $\varepsilon_k = \frac{\varepsilon}{2^k}$

Probability of leaving $\Psi_0$:

$\leq \sum_{i=1}^{\infty} \varepsilon_i = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon$

$\forall \varepsilon < \frac{1}{2}$. 
GENERALIZED LOCAL WITNESSES
FOR PLAYER 1 IN $C_0 \otimes (0,1)$ GAMES

$\Psi : \mathcal{V} \to \mathbb{N} \cup \{\infty\}$

FACT PLAYER 1 WINS
WITH PROBABILITY $> 0$

PROOF (INDUCTION ON RANK)

- RANK $T$ IS EVEN

$\alpha :$ IN $k$-TH STEP PLAY $\alpha_{\varepsilon_k}$

s.t. $\mathbb{P}^{\alpha_{\varepsilon_k}}[\Psi_{<\infty}] > 0 \quad \star$

$\quad$ OR $\quad \mathbb{P}^{\alpha_{\varepsilon_k}}[\Psi_{>\infty}] \leq \varepsilon_k$

IF $\star$ NEVER HOLDS THEN
PROBABILITY OF INCREASING RANK IS

$\leq \sum_{i=1}^{\infty} \varepsilon_i = \varepsilon$
TRANSLATING THE \( \exists \varepsilon > 0. \ \forall \varepsilon \) CONDITION

\[ \exists \varepsilon > 0. \ \exists \alpha. \ \forall \beta. \ P^\alpha_\beta[\varphi_{\infty}] = 0 \land P^\alpha_\beta[\varphi_{<\infty}] > \varepsilon \]

\[ \alpha : \begin{bmatrix} A & \frac{1}{|A|} \\ \bar{A} & 0 \end{bmatrix} \quad \text{s.t.} \quad \forall \alpha. \ \exists \alpha \in A. \ P^{\alpha}_\beta[\varphi_{<\infty}] > 0 \]

\[ \implies P^\alpha_\beta[\varphi_{<\infty}] \geq \frac{\mu}{|A|} \]

\[ \mu = \text{THE SMALLEST NON-ZERO } P^\alpha_\beta[\varphi_{\infty}] \]
TRANSLATING THE \[ \exists \varepsilon > 0. \frac{\dot{P}}{P \varepsilon} \] CONDITION

\[ \exists \varepsilon > 0. \exists \alpha. \forall \beta. P_{v}^{ab}[\psi_{\infty}] = 0 \land P_{v}^{ab}[\psi_{<\infty}] > \varepsilon \]

\[ \forall b. \exists a. (\forall b'. P_{v}^{a b'}[\psi_{\infty}] = 0) \land P_{v}^{a b}[\psi_{<\infty}] > 0 \]

\[
\begin{array}{c}
1 \quad 1 \quad 1 \\
\vdots \quad \vdots \quad \vdots \\
v \quad \quad \quad \quad \quad b' \quad b \\
\downarrow \quad \quad \quad \quad \quad \downarrow \quad \downarrow \\
P_{v}^{a b}[\psi_{<\infty}] > 0 \quad P_{v}^{a b'}[\psi_{\infty}] = 0 \\
\end{array}
\]

<table>
<thead>
<tr>
<th>( P_{v}^{a b}[\psi_{&lt;\infty}] &gt; 0 ) ?</th>
<th>( P_{v}^{a b}[\psi_{\infty}] = 0 ) ?</th>
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<td>( \delta(v, a, b) )</td>
<td>( \delta(v, a, b') )</td>
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The \( \forall \epsilon < \frac{1}{2}, \text{ or } \epsilon \geq \frac{1}{2} \) is dual to \( \exists \epsilon > 0. \text{ or } \exists \epsilon \leq \frac{1}{2} \).

\( \forall \epsilon < \frac{1}{2}, \exists \beta, \forall \alpha, P^a_{\alpha} [\psi_{<\alpha}] > 0 \text{ or } P^a_{\beta} [\psi_{>\beta}] \leq \epsilon \)

\[
\begin{align*}
P^a_{\alpha} [\psi_{<\alpha}] &= 0 \\
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GENERALIZED LOCAL WITNESSES FOR CONCURRENT CO-BÜCHI GAMES

PLAYER 0

\( \varphi : V \rightarrow \mathbb{N} \cup \{\infty\} \)

\[ \exists \varepsilon > 0. \]

PLAYER 0 WINS WITH PROBABILITY 1
GENERALIZED LOCAL WITNESSES
FOR CONCURRENT CO-BÜCHI GAMES

PLAYER 1

\[ \psi : \nu \rightarrow \mathbb{N}^2 \cup \{\infty\} \]

\[ \exists m \geq 1. \forall \varepsilon \leq \frac{1}{2}. \]

\[ \pi_i, \leq \varepsilon \cdot \pi_i^< \]
TRANSFORMING THE \( P^i / P^e \) OR \( P^i / P^e \) AND \( P^i / P^e \) CONDITION

PLAYER 0: \( \exists \epsilon > 0. \exists x. \forall \beta. P^u_{\alpha^u}[\psi_{x,\epsilon}] = 1 \land \land P^u_{\alpha^u}[\psi_{x,\epsilon}] > \epsilon \cdot P^u_{\alpha^u}[\psi_{x,\epsilon}] \)

PLAYER 1: \( \exists m. \forall \epsilon < \frac{1}{2}. \exists \beta. \forall \alpha. P^u_{\alpha^u}[\psi_{\epsilon^u}] > 0 \lor \lor (P^u_{\alpha^u}[\psi_{\epsilon^u}] > \epsilon^m \land P^u_{\alpha^u}[\psi_{\epsilon^u}] \leq \epsilon \cdot P^u_{\alpha^u}[\psi_{\epsilon^u}]) \)
To be done

- Translations of $C(1,d)$ and $C(0,d)$ games to turn-based $(0,d)$ games
  $\Rightarrow$ Improving complexity from $O(n^d)$ to $O(n^d)$

- Same for limit-sure win

- Generalization to quantitative solution of concurrent probabilistic games?
  $\Rightarrow$ Algorithms?