Kronecker products and iterated matrix multiplication

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Motivation

- Valiant (1979): determinant vs permanent question: Given m, what is the smallest n such that $\exists y_{i,j} \in \mathbb{F} \cup \{x_{i,j}\}$ with

$$\det \begin{pmatrix} y_{1,1} & \cdots & y_{1,n} \\ \vdots & \ddots & \vdots \\ y_{n,1} & \cdots & y_{n,n} \end{pmatrix} = \underbrace{\sum_{\pi \in \mathfrak{S}_m} \prod_{i=1}^{m} x_{i,\pi(i)}}_{=:\mathrm{per}_m} \qquad \det \begin{pmatrix} 0 & x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & x_{22} & x_{23} & 0 \\ 0 & 0 & 1 & 0 & x_{21} & 0 & x_{23} \\ 0 & 0 & 0 & 1 & 0 & x_{21} & x_{22} \\ x_{33} & 0 & 0 & 0 & 1 & 0 & 0 \\ x_{32} & 0 & 0 & 0 & 0 & 1 & 0 \\ x_{31} & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \mathrm{per}_3.$$

- Conj.: n grows superpolynomially in m.
- Known to be false in characteristic 2 because $\mathrm{per}_m = \det_m$. Hence, cannot be proved via field-independent methods
- Replacements per_m that work over every field come from combinatorics (Hamiltonian cycles), not from algebra
- Mulmuley–Sohoni (2001): Geometric complexity theory (GCT) to resolve det vs per
- Kadish–Landsberg (2012), Bürgisser–I–Panova (2016): Some GCT conjectures are false, since $\deg(\det_n) \neq \deg(\mathrm{per}_m)$
- Approaches via the width of polynomials have some issues (Gesmundo 2015 + my AGATES Warsaw talk 2022)

> We fix this. We abandon determinantal complexity.
> • Works over all commutative semirings. • Works for polynomials, tensors, and exterior algebra.
> (this is exactly the level of generality in `mathlib`)

Nice corollaries of our work (unintended):

- Generalization of the Cayley–Hamilton theorem, new record ABP construction for the determinant (Mahajan–Vinay 1997), answer a question from Mahajan–Vinay (1999) about combinatorial algorithms for the determinant
- Gurvits' (2004, 2005) VNP-hardness results for the mixed discriminant and Cayley's first hyperdeterminant hold over every commutative ring (Gurvits had fields of char $\neq 2$). Also higher order permanents and higher order hyperpfaffians.
- Christandl–Fawzi–Ta–Zuiddam (2022) asymptotic spectrum works for polynomials in arbitrary characteristic

Let

$$X_m^{(k)} := \begin{pmatrix} x_{1,1}^{(k)} & \cdots & x_{1,m}^{(k)} \\ \vdots & \ddots & \vdots \\ x_{m,1}^{(k)} & \cdots & x_{m,m}^{(k)} \end{pmatrix}.$$

For fixed m and d, determine the smallest n such that

$$\left(X_m^{(1)} \cdots X_m^{(d)}\right)^{\boxtimes 2} \qquad \leq \qquad X_n^{(1)} \cdots X_n^{(d)}$$

For fixed d, the smallest n grows polynomially in m, i.e., $n = O(m^{\xi_d + \varepsilon})$.

Conjecture: The sequence $(\xi_d)_{d\in\mathbb{N}}$ is unbounded.

Implies Valiant's conjecture, and is implied by Bläser–Engels' conjecture.

Let f be a homogeneous degree d polynomial.

The **Waring rank** $\mathrm{WR}(f)$ is defined as the smallest r such that $\exists$ linear forms with

$$\begin{aligned} f &= (\ell_1\ \ell_2\ \cdots\ \ell_r) \begin{pmatrix} \ell_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r \end{pmatrix} \begin{pmatrix} \ell_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r \end{pmatrix} \cdots \begin{pmatrix} \ell_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r \end{pmatrix} \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_r \end{pmatrix} \\ &= \operatorname{tr}\left(\begin{pmatrix} \ell_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r \end{pmatrix} \cdots \begin{pmatrix} \ell_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r \end{pmatrix} \right) \end{aligned}$$

The **Chow rank** $\mathrm{CR}(f)$ is defined as the smallest r such that $\exists$ linear forms with

$$\begin{aligned} f &= (\ell_1^{(1)}\ \ell_2^{(1)}\ \cdots\ \ell_r^{(1)}) \begin{pmatrix} \ell_1^{(2)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r^{(2)} \end{pmatrix} \begin{pmatrix} \ell_1^{(3)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r^{(3)} \end{pmatrix} \cdots \begin{pmatrix} \ell_1^{(d-1)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r^{(d-1)} \end{pmatrix} \begin{pmatrix} \ell_1^{(d)} \\ \vdots \\ \ell_r^{(d)} \end{pmatrix} \\ &= \operatorname{tr}\left(\begin{pmatrix} \ell_1^{(1)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r^{(1)} \end{pmatrix} \cdots \begin{pmatrix} \ell_1^{(d)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r^{(d)} \end{pmatrix} \right) \end{aligned}$$

Mateusz Michałek on Tuesday: "There is nothing stopping us from dropping the word *diagonal*."
So we now fill **all** matrix entries with linear forms!

Two variants:

- The smallest r is called the **width**:

$$f = (\ell^{(1)}_{1,1}\ \ell^{(1)}_{1,2}\ \cdots\ \ell^{(1)}_{1,r}) \begin{pmatrix} \ell^{(2)}_{1,1} & \cdots & \ell^{(2)}_{1,r} \\ \vdots & \ddots & \vdots \\ \ell^{(2)}_{r,1} & \cdots & \ell^{(2)}_{r,r} \end{pmatrix} \begin{pmatrix} \ell^{(3)}_{1,1} & \cdots & \ell^{(3)}_{1,r} \\ \vdots & \ddots & \vdots \\ \ell^{(3)}_{r,1} & \cdots & \ell^{(3)}_{r,r} \end{pmatrix} \cdots \begin{pmatrix} \ell^{(d-1)}_{1,1} & \cdots & \ell^{(d-1)}_{1,r} \\ \vdots & \ddots & \vdots \\ \ell^{(d-1)}_{r,1} & \cdots & \ell^{(d-1)}_{r,r} \end{pmatrix} \begin{pmatrix} \ell^{(d)}_{1,1} \\ \vdots \\ \ell^{(d)}_{1,r} \end{pmatrix}$$

- The smallest r is the width's little cousin ($\leq w(f)$):

$$f = \operatorname{tr}\left(\begin{pmatrix} \ell^{(1)}_{1,1} & \cdots & \ell^{(1)}_{1,r} \\ \vdots & \ddots & \vdots \\ \ell^{(1)}_{r,1} & \cdots & \ell^{(1)}_{r,r} \end{pmatrix} \cdots \begin{pmatrix} \ell^{(d)}_{1,1} & \cdots & \ell^{(d)}_{1,r} \\ \vdots & \ddots & \vdots \\ \ell^{(d)}_{r,1} & \cdots & \ell^{(d)}_{r,r} \end{pmatrix} \right)$$

- Taking traces **looks** nice, but:
  the noncommutative variant is not Zariski-closed, and **CAVEAT** when working field-independently!

Iterated matrix multiplication

For $f \in \mathrm{Sym}^d V \cup \bigwedge^d V \cup \bigotimes^d V$, the **width** $w(f)$ is defined as the smallest n such that $\exists$homog. lin. $\ell_{i,j}^{(k)}$ with

$$f = \begin{pmatrix} \ell_{1,1}^{(1)} & \cdots & \ell_{1,n}^{(1)} \end{pmatrix} \begin{pmatrix} \ell_{1,1}^{(2)} & \cdots & \ell_{1,n}^{(2)} \\ \vdots & \ddots & \vdots \\ \ell_{n,1}^{(2)} & \cdots & \ell_{n,n}^{(2)} \end{pmatrix} \begin{pmatrix} \ell_{1,1}^{(3)} & \cdots & \ell_{1,n}^{(3)} \\ \vdots & \ddots & \vdots \\ \ell_{n,1}^{(3)} & \cdots & \ell_{n,n}^{(3)} \end{pmatrix} \cdots \begin{pmatrix} \ell_{1,1}^{(d-1)} & \cdots & \ell_{1,n}^{(d-1)} \\ \vdots & \ddots & \vdots \\ \ell_{n,1}^{(d-1)} & \cdots & \ell_{n,n}^{(d-1)} \end{pmatrix} \begin{pmatrix} \ell_{1,1}^{(d)} \\ \ell_{2,1}^{(d)} \\ \vdots \\ \ell_{n,1}^{(d)} \end{pmatrix}$$

Nisan's 1990 result for $f \in \bigotimes^d V$

$$w(f) = \max\{\mathsf{rk}(\mathcal{F}_j(f)) \mid 1 \leq j \leq d-1\}$$

where $\mathcal{F}_j(f)$ is f interpreted as a matrix in $(\bigotimes^j V) \otimes (\bigotimes^{d-j} V)$.
Forbes' observation 2016: For tensors we have width = border-width.

This does not hold when working with the trace instead.

This definition works for arbitrary commutative semirings.

Bläser–Dörfler–I (2020), "De-bordering": Over $\mathbb{C}$, for a homogeneous polynomial f, we have $\underline{\mathrm{WR}}(f) \geq w(f)$.

Bürgisser–Cucker–Lairez (2020): width to describe structured systems of polynomial eqns, and solve Smale's 17th problem.

Restrictions

- $w(f)$ is defined as the smallest n such that $\exists$ homog. lin. $\ell_{i,j}^{(k)}$ such that
$$f = \begin{pmatrix} \ell_{1,1}^{(1)} & \cdots & \ell_{1,n}^{(1)} \end{pmatrix} \begin{pmatrix} \ell_{1,1}^{(2)} & \cdots & \ell_{1,n}^{(2)} \\ \vdots & \ddots & \vdots \\ \ell_{n,1}^{(2)} & \cdots & \ell_{n,n}^{(2)} \end{pmatrix} \begin{pmatrix} \ell_{1,1}^{(3)} & \cdots & \ell_{1,n}^{(3)} \\ \vdots & \ddots & \vdots \\ \ell_{n,1}^{(3)} & \cdots & \ell_{n,n}^{(3)} \end{pmatrix} \cdots \begin{pmatrix} \ell_{1,1}^{(d-1)} & \cdots & \ell_{1,n}^{(d-1)} \\ \vdots & \ddots & \vdots \\ \ell_{n,1}^{(d-1)} & \cdots & \ell_{n,n}^{(d-1)} \end{pmatrix} \begin{pmatrix} \ell_{1,1}^{(d)} \\ \ell_{2,1}^{(d)} \\ \vdots \\ \ell_{n,1}^{(d)} \end{pmatrix}$$
- We write $f \leq h$ if $f = h(\ell_1, \ldots, \ell_N)$ for homogeneous linear ℓ_i.
$$\Gamma_{n,d} := \begin{pmatrix} x_{1,1}^{(1)} & \cdots & x_{1,n}^{(1)} \end{pmatrix} \begin{pmatrix} x_{1,1}^{(2)} & \cdots & x_{1,n}^{(2)} \\ \vdots & \ddots & \vdots \\ x_{n,1}^{(2)} & \cdots & x_{n,n}^{(2)} \end{pmatrix} \begin{pmatrix} x_{1,1}^{(3)} & \cdots & x_{1,n}^{(3)} \\ \vdots & \ddots & \vdots \\ x_{n,1}^{(3)} & \cdots & x_{n,n}^{(3)} \end{pmatrix} \cdots \begin{pmatrix} x_{1,1}^{(d-1)} & \cdots & x_{1,n}^{(d-1)} \\ \vdots & \ddots & \vdots \\ x_{n,1}^{(d-1)} & \cdots & x_{n,n}^{(d-1)} \end{pmatrix} \begin{pmatrix} x_{1,1}^{(d)} \\ x_{2,1}^{(d)} \\ \vdots \\ x_{n,1}^{(d)} \end{pmatrix}$$

$w(f) \leq n$ iff $f \leq \Gamma_{n,d}$

For $f \in \mathrm{Sym}^d \mathbb{C}^N$ let
$$\mathrm{stab}(f) = \{g \in \mathsf{GL}_N \mid gf = f\}.$$

These all have reductive stabilizers: $x_1 x_2 \cdots x_d$, $x_1^d + x_2^d + \cdots + x_d^d$, $\det_n$, per_m

BUT $\mathrm{stab}(\Gamma_{n,d})$ is not reductive!

$$\Gamma_{n,d} = \begin{pmatrix} x^{(1)}_{1,1} & \cdots & x^{(1)}_{1,r} \end{pmatrix} \begin{pmatrix} x^{(2)}_{1,1} & \cdots & x^{(2)}_{1,r} \\ \vdots & \ddots & \vdots \\ x^{(2)}_{r,1} & \cdots & x^{(2)}_{r,r} \end{pmatrix} \begin{pmatrix} x^{(3)}_{1,1} & \cdots & x^{(3)}_{1,r} \\ \vdots & \ddots & \vdots \\ x^{(3)}_{r,1} & \cdots & x^{(3)}_{r,r} \end{pmatrix} \cdots \begin{pmatrix} x^{(d-1)}_{1,1} & \cdots & x^{(d-1)}_{1,r} \\ \vdots & \ddots & \vdots \\ x^{(d-1)}_{r,1} & \cdots & x^{(d-1)}_{r,r} \end{pmatrix} \begin{pmatrix} x^{(d)}_{1,1} \\ x^{(d)}_{2,1} \\ \vdots \\ x^{(d)}_{r,1} \end{pmatrix}$$

$$\Xi_{n,d} := \begin{pmatrix} x^{(1)}_{1,1} & \cdots & x^{(1)}_{1,n} \\ \vdots & \ddots & \vdots \\ x^{(1)}_{n,1} & \cdots & x^{(1)}_{n,n} \end{pmatrix} \begin{pmatrix} x^{(2)}_{1,1} & \cdots & x^{(2)}_{1,n} \\ \vdots & \ddots & \vdots \\ x^{(2)}_{n,1} & \cdots & x^{(2)}_{n,n} \end{pmatrix} \cdots \begin{pmatrix} x^{(d)}_{1,1} & \cdots & x^{(d)}_{1,n} \\ \vdots & \ddots & \vdots \\ x^{(d)}_{n,1} & \cdots & x^{(d)}_{n,n} \end{pmatrix}$$

	Zariski-closed End-orbit	reductive stabilizer
$\Gamma_{n,d}$	✓	☹
$\mathrm{tr}(\Xi_{n,d})$	☹	✓
$\Xi_{n,d}$	✓	✓

$$\Xi_{n,d} \in \mathbb{C}^n \otimes \mathrm{Sym}^d \mathbb{C}^{n^2 d} \otimes \mathbb{C}^{n*}$$

$$\mathrm{stab}_{\mathsf{GL}_n \times \mathsf{GL}_{n^2 d} \times \mathsf{GL}_n}(\Xi_{n,d}) = \mathsf{GL}_n^{d+1}$$

- $x_1^d + x_2^d = \prod_{k=0}^{d-1}(x_1 - \zeta_k x_2)$, where $\zeta_k = e^{(2k+1)\pi i/d}$
- $w(x_1^d + \cdots + x_n^n) \;\leq\; \lceil \frac{n}{2} \rceil$
- Kumar 2019: $w(x_1^d + \cdots + x_n^n) \;=\; \lceil \frac{n}{2} \rceil$
- Easy to see:

$$w\left(\begin{pmatrix} x_1^d & 0 & \dots & 0 \\ 0 & x_2^d & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & x_n^d \end{pmatrix}\right) = n.$$

Gradient and Hessian

Giorgio Ottaviani mentioned yesterday:

- $\nabla(h \boxtimes f) = \nabla(h) \boxtimes \nabla(f)$
- $\mathrm{Hess}(h \boxtimes f) = \mathrm{Hess}(h) \boxtimes \mathrm{Hess}(f)$

Computing Determinants

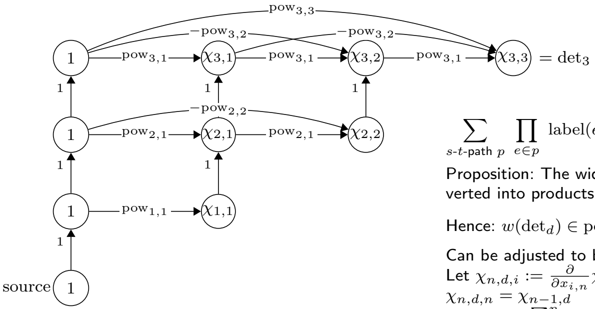
$$X_n := \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \qquad \det(X_n + \mathsf{diag}(t,\ldots,t)) = \sum_{d>0} \chi_{n,d} \cdot t^{n-d}$$

Strengthening of the Cayley–Hamilton theorem

For all $n, d \in \mathbb{N}$ we have $(\nabla \chi_{n,d+1})^T \ = \ \sum_{i=0}^{d} (-1)^i \, \chi_{n,d-i} \, (X_n)^i.$

Take the bottom-right entry: $\mathrm{pow}_{n,i} := [(X_n)^i]_{n,n}$ $\qquad \chi_{n-1,d} \ = \ \sum_{i=0}^{d} (-1)^i \, \chi_{n,d-i} \, \mathrm{pow}_{n,i}$. Rearrange:

$$\chi_{n,d} \ = \ \chi_{n-1,d} + \sum_{i=1}^{d} (-1)^{i+1} \, \chi_{n,d-i} \, \mathrm{pow}_{n,i}.$$



$$\sum_{s\text{-}t\text{-path } p} \ \prod_{e \in p} \mathrm{label}(e) = \det_d$$

Proposition: The width of such graph constructions can be converted into products of matrices of the same width.

Hence: $w(\det_d) \in \mathrm{poly}(d)$

Can be adjusted to beat [Mahajan–Vinay 1997]:
Let $\chi_{n,d,i} := \frac{\partial}{\partial x_{i,n}} \chi_{n,d+1}$.
$\chi_{n,d,n} = \chi_{n-1,d}$
$\chi_{n,d,i} = -\sum_{j=1}^{n} \chi_{n,d-1,j} \cdot x_{j,i}$ for $i < n$.

- Tensor product $\otimes$ increases the tensor order:
  - $\mathbb{F}^n \otimes \mathbb{F}^m = \mathbb{F}^{n \times m}$ matrices.
  - $\mathbb{F}^n \otimes \mathbb{F}^m \otimes \mathbb{F}^\ell = \mathbb{F}^{n \times m \times \ell}$ order 3 tensors.
- Kronecker product $\boxtimes$ preserves the tensor order:
  - $\mathbb{F}^n \boxtimes \mathbb{F}^m = \mathbb{F}^{nm}$.
  - $\mathbb{F}^{n_1 \times n_2} \boxtimes \mathbb{F}^{m_1 \times m_2} = \mathbb{F}^{(n_1 m_1) \times (n_2 m_2)}$:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \qquad A \boxtimes B = \begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix} = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}$$

  - $\boxtimes$ works for every fixed degree d.
  - $\boxtimes$ = "$\otimes$ and then flatten"

Kronecker product and the matrix multiplication exponent

- The matrix multiplication tensor

$$M_n = \sum_{i,j,k\in\{1,\dots,n\}} (e_i \boxtimes e_j) \otimes (e_j \boxtimes e_k) \otimes (e_k \boxtimes e_i) \qquad \in \mathbb{F}^{n^2} \otimes \mathbb{F}^{n^2} \otimes \mathbb{F}^{n^2}$$

- $M_n \boxtimes M_m = M_{nm}$
- $M_n \boxtimes M_m = M_{nm}$ was first used by Strassen (1969) for subcubic matrix multiplication algorithms.
- Used today for the Coppersmith–Winograd tensor to obtain the fastest matrix multiplication algorithms.
- Many tasks in computational linear algebra run in roughly the same time:
  - matrix multiplication
  - solving systems of linear equations
  - determinant
  - matrix inverse
  - LU decomposition
  - etc.

$$\Xi_{n,d} := \begin{pmatrix} x^{(1)}_{1,1} & \cdots & x^{(1)}_{1,n} \\ \vdots & \ddots & \vdots \\ x^{(1)}_{n,1} & \cdots & x^{(1)}_{n,n} \end{pmatrix} \begin{pmatrix} x^{(2)}_{1,1} & \cdots & x^{(2)}_{1,n} \\ \vdots & \ddots & \vdots \\ x^{(2)}_{n,1} & \cdots & x^{(2)}_{n,n} \end{pmatrix} \cdots \begin{pmatrix} x^{(d)}_{1,1} & \cdots & x^{(d)}_{1,n} \\ \vdots & \ddots & \vdots \\ x^{(d)}_{n,1} & \cdots & x^{(d)}_{n,n} \end{pmatrix}$$

This section: Define $\Xi_{n,d} \boxtimes \Xi_{n,d}$

Kronecker product of polynomials in characteristic 0

- Polarization $\mathrm{polar} : \mathbb{C}[x_1, \dots, x_N]_d \to (\bigotimes^d \mathbb{C}^N)^{\mathfrak{S}_d}$
- Restitution $\mathrm{resti} : \bigotimes^d \mathbb{C}^N \to \mathbb{C}[x_1, \dots, x_N]_d$

Definition in char 0

Let $h, f \in \mathbb{C}[x_1, \dots, x_N]_d$.

$$h \boxtimes f \;:=\; \tfrac{1}{d!}\mathrm{resti}(\mathrm{polar}(h) \boxtimes \mathrm{polar}(f))$$

$\mathrm{mon}_d = x_1 x_2 \cdots x_d$.

- The permanent polynomial: $\mathrm{mon}_d \boxtimes \mathrm{mon}_d$.
- The hyperpermanent: $\mathrm{mon}_d^{\boxtimes m}$.
- The rectangular permanent: $\mathrm{mon}_d \boxtimes e_{n,d}$.
- The mixed discriminant: $\det_d \boxtimes \mathrm{mon}_d$.
- Cayley's first hyperdeterminant: $\det_d^{\boxtimes m}$.

$\boxtimes$ over commutative semirings R

- Let $\mathrm{Sym}_d(V) := (\bigotimes^d V)^{\mathfrak{S}_d}$ (symmetric tensors).
  Let $\mathrm{Sym}^d(V) := (\bigotimes^d V)/\mathfrak{S}_d$ (coinvariants = polynomials)
- R-bilinear map $\boxtimes : \bigotimes^d V \times \bigotimes^d W \to \bigotimes^d (V \boxtimes W)$.
- Restrict to $\psi : \mathrm{Sym}_d(V) \times \bigotimes^d W \to \bigotimes^d (V \boxtimes W)$.
- Fix $s \in \mathrm{Sym}_d(V)$.
- $\mathfrak{S}_d$-equivariant R-linear map $\phi_s : \bigotimes^d W \to \bigotimes^d (V \boxtimes W)$, $f \mapsto \psi(s, f)$
- Apply coinvariant functor: $\psi_s : \mathrm{Sym}^d(W) \to \mathrm{Sym}^d(V \boxtimes W)$, $[f] \mapsto [\psi(s, f)]$.
- R-linear map $\kappa : \mathrm{Sym}_d(V) \to \mathrm{Hom}\big(\mathrm{Sym}^d(W), \mathrm{Sym}^d(V \boxtimes W)\big)$, $\qquad s \mapsto \psi_s$
- Altogether: R-bilinear map

$$\boxtimes : \mathrm{Sym}_d(V) \times \mathrm{Sym}^d(W) \to \mathrm{Sym}^d(V \boxtimes W), \qquad (s, [f]) \mapsto \psi_s([f]) = [\psi(s, f)]$$

Definition (Kronecker product of polynomials)

Let $h, f \in R[x_1, \ldots, x_N]_d$. $\qquad h \boxtimes f \;:=\; \mathrm{resti}(\mathrm{polar}(h) \boxtimes f)$. $\qquad$ ($h \boxtimes f = f \boxtimes h$ up to renaming variables)

On monomials: $(x_1 \cdots x_d) \boxtimes (y_1 \cdots y_d) = \sum_{\pi \in \mathfrak{S}_d} (x_1 \boxtimes y_{\pi(1)}) \cdots (x_d \boxtimes y_{\pi(d)})$.

The hypercomputant $\mathsf{h}\Xi_{n,d}$

$$\mathsf{h}\Xi_{n,d} \;:=\; \Xi_{n,d} \boxtimes \Xi_{n,d}$$

an $n^2 \times n^2$ matrix of homog. degree d polynomials. Its bottom-right entry: $\mathsf{h}\Gamma_{n,d} \;=\; \Gamma_{n,d} \boxtimes \Gamma_{n,d}$

Hamiltonian cycle polynomial: $\mathrm{HC}_d \;:=\; \sum_{\phi \in \mathfrak{S}_{d-1}} x^{(1)}_{d,\phi(1)} \cdot x^{(2)}_{\phi(1),\phi(2)} \cdots x^{(d)}_{\phi(d-1),d}$

Theorem: $\mathrm{HC}_d \leq \mathsf{h}\Gamma_{d,d}$.

Proof

$$\Gamma_{d,d} \;=\; \sum_{\phi:[d-1]\to[d]} x^{(1)}_{d,\phi(1)} \cdot x^{(2)}_{\phi(1),\phi(2)} \cdot x^{(3)}_{\phi(2),\phi(3)} \cdots x^{(d)}_{\phi(d-1),d}.$$

$$\Gamma_{d,d} \boxtimes \Gamma_{d,d} \;\geq\; \mathrm{mon}_d \boxtimes \Gamma_{d,d} \;=\; \sum_{\substack{\phi:[d-1]\to[d] \\ \pi \in \mathfrak{S}_d}} x^{(1)}_{\pi(1),d,\phi(1)} \cdot x^{(2)}_{\pi(2),\phi(1),\phi(2)} \cdots x^{(d)}_{\pi(d),\phi(d-1),d}.$$

Define $T(x^{(k)}_{a,b,c}) = x^{(k)}_{b,c}$ if $a = c$, and $T(x^{(k)}_{a,b,c}) = 0$ otherwise. We calculate

$$T(\mathrm{mon}_d \boxtimes \Gamma_{d,d}) = \sum_{\substack{\phi \in \mathfrak{S}_d \\ \phi(d)=d}} x^{(1)}_{d,\phi(1)} \cdot x^{(2)}_{\phi(1),\phi(2)} \cdots x^{(d)}_{\phi(d-1),d} \;=\; \mathrm{HC}_d. \qquad \square$$

With a small argument, this lifts two hardness results of Gurvits (2004, 2005) to all commutative rings:

$$\mathrm{HC}_d \quad \leq \quad \mathrm{mon}_d \boxtimes \Gamma_{d,d} \quad \leq_{\mathrm{p}} \quad \mathrm{mon}_d \boxtimes \det_d \quad \leq \quad \det_d \boxtimes \det_d$$

Exponents

$$\xi_d \ := \ \inf\{e \mid w(\mathsf{h}\Xi_{n,d}) \in O(n^e)\},$$

d	2	3	4	5	6	7			
ξ_d for tensors over $\mathbb{C}$	4	4	6	6	8	8	$\cdots$	$\Theta(d)$	(Nisan)
ξ_d for polynomials over $\mathbb{R}_{\geq 0}$	4	4	6	6	6	8	$\cdots$	$\Theta(d)$	(pathwidth)

Main open question

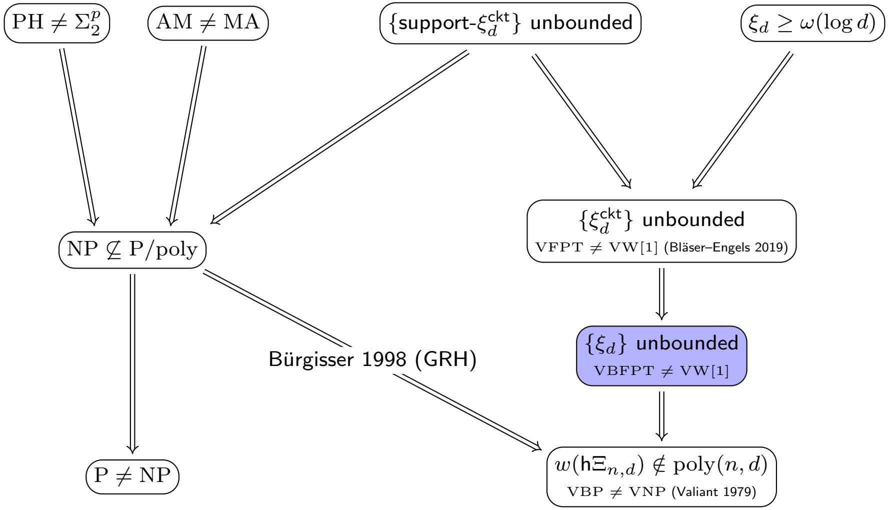
Is the set of exponents $\{\xi_d \mid d \in \mathbb{N}\}$ unbounded for polynomials over $\mathbb{C}$?

$\xi_d \geq 2$. Best known algorithm in the general case is **monotone**.

Theorem (Chatterjee–Kumar–Volk 2024): Determinantal complexity gives the **same** exponent, over every field.

Theorem (Ahmed Shaaban 2026 using GPT 5.6):
There is a commutative ring where determinantal complexity gives a different exponent.

For polynomials over $\mathbb{C}$:



Summary

- Guided by fixing some rough edges in the definitions of geometric complexity theory (GCT).
- Instead of polynomials, study **double-index** (n and d) **matrices** of polynomials.
- This gives a short existence proof for polysized ABPs for $\det_d$, gives a new record ABP for $\det_d$, and answers a question by Mahajan–Vinay.
- Both properties at the same time: The stabilizer of $\Xi_{n,d}$ is reductive and the tensor variant is Zariski-closed.
- The definition of $\boxtimes$ for polynomials solves issues with the characteristic.
- The equivariance of $\boxtimes$ gives a short way for lifting two hardness results of Gurvits to all commutative rings.
- And possibly more ...

Thank you for your attention!