# Kronecker products and iterated matrix multiplication


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| Valiant 1979<br>Mulmuley–Sohoni 2001 | Determinant<br>$\sum_{\pi \in \mathfrak{S}_n} \mathrm{sgn}(\pi) \prod_{i=1}^{n} x_{i,\pi(i)}$ | Permanent<br>$\sum_{\pi \in \mathfrak{S}_n} \prod_{i=1}^{n} x_{i,\pi(i)}$ |

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| This work | Iterated matrix multiplication<br>(="computant") | hypercomputant |

1. Motivation
2. Iterated matrix multiplication: the computant
3. Kronecker products: The hypercomputant

## Motivation

- Valiant (1979): determinant vs permanent question: Given $m$, what is the smallest $n$ such that $\exists y_{i,j} \in \mathbb{F} \cup \{x_{i,j}\}$ with

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- Works in geometric complexity: Reductive connected stabilizer, and tensor variant Zariski-closed.

Nice unintended results (serve as sanity checks that this is a good direction):

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- Christandl–Fawzi–Ta–Zuiddam (2022) asymptotic spectrum works for homogeneous polynomials in arbitrary characteristic

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$$X_m^{(k)} := \begin{pmatrix} x_{1,1}^{(k)} & \cdots & x_{1,m}^{(k)} \\ \vdots & \ddots & \vdots \\ x_{m,1}^{(k)} & \cdots & x_{m,m}^{(k)} \end{pmatrix}.$$

For fixed $m$ and $d$, determine the smallest $n$ such that

$$\underbrace{\left(X_m^{(1)} \cdots X_m^{(d)}\right)^{\boxtimes 2}}_{\text{hypercomputant}} \qquad \leq \qquad \underbrace{X_n^{(1)} \cdots X_n^{(d)}}_{\text{computant}}$$

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Conjecture: The set $\{\xi_d \mid d \in \mathbb{N}\}$ is unbounded.

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$$\underset{\text{Bläser–Engels conjecture}}{\mathrm{VFPT} \neq \mathrm{VW}[1]} \implies \underset{\{\xi_d | d \in \mathbb{N}\} \text{ unbounded}}{\mathrm{VBFPT} \neq \mathrm{VW}[1]} \implies \underset{\text{Valiant's conjecture}}{\mathrm{VBP} \neq \mathrm{VNP}}$$

Let $f$ be a homogeneous degree $d$ polynomial.

The **Waring rank** $\mathrm{WR}(f)$ is defined as the smallest $r$ such that $\exists$ linear forms with

$$f \quad = \quad (\ell_1\ \ell_2\ \cdots\ \ell_r)\begin{pmatrix}\ell_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \ell_r\end{pmatrix}\begin{pmatrix}\ell_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \ell_r\end{pmatrix}\cdots\begin{pmatrix}\ell_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \ell_r\end{pmatrix}\begin{pmatrix}\ell_1\\ \vdots\\ \ell_r\end{pmatrix}$$

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We now fill **all** matrix entries with linear forms!

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We now fill **all** matrix entries with linear forms!
This gives two variants.

Two variants:

- The smallest $r$ is called the **width**:

$$f = (\ell_{1,1}^{(1)}\ \ell_{1,2}^{(1)}\ \cdots\ \ell_{1,r}^{(1)}) \begin{pmatrix} \ell_{1,1}^{(2)} & \cdots & \ell_{1,r}^{(2)} \\ \vdots & \ddots & \vdots \\ \ell_{r,1}^{(2)} & \cdots & \ell_{r,r}^{(2)} \end{pmatrix} \begin{pmatrix} \ell_{1,1}^{(3)} & \cdots & \ell_{1,r}^{(3)} \\ \vdots & \ddots & \vdots \\ \ell_{r,1}^{(3)} & \cdots & \ell_{r,r}^{(3)} \end{pmatrix} \cdots \begin{pmatrix} \ell_{1,1}^{(d-1)} & \cdots & \ell_{1,r}^{(d-1)} \\ \vdots & \ddots & \vdots \\ \ell_{r,1}^{(d-1)} & \cdots & \ell_{r,r}^{(d-1)} \end{pmatrix} \begin{pmatrix} \ell_{1,1}^{(d)} \\ \vdots \\ \ell_{1,r}^{(d)} \end{pmatrix}$$

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- The smallest $r$ is the width's cousin:

$$f = \mathrm{tr}\left( \begin{pmatrix} \ell^{(1)}_{1,1} & \cdots & \ell^{(1)}_{1,r} \\ \vdots & \ddots & \vdots \\ \ell^{(1)}_{r,1} & \cdots & \ell^{(1)}_{r,r} \end{pmatrix} \cdots \begin{pmatrix} \ell^{(d)}_{1,1} & \cdots & \ell^{(d)}_{1,r} \\ \vdots & \ddots & \vdots \\ \ell^{(d)}_{r,1} & \cdots & \ell^{(d)}_{r,r} \end{pmatrix} \right)$$

  studied by Gesmundo (2016)

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- Taking traces **looks** nice, but:
  the noncommutative variant is not Zariski-closed (see next slide), and **CAVEAT** when working field-independently!

# Iterated matrix multiplication

For $f \in \mathrm{Sym}^d V \cup \bigwedge^d V \cup \bigotimes^d V$, the **width** $w(f)$ is defined as the smallest $n$ such that $\exists$homog. lin. $\ell_{i,j}^{(k)}$ with

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## Nisan's 1990 result for $f \in \bigotimes^d V$

$$w(f) = \max\{\mathsf{rk}(\mathcal{F}_j(f)) \mid 1 \le j \le d-1\}$$

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We fully understand the width "up to the action of the symmetric group".

Bürgisser–Cucker–Lairez (2020): width to describe structured systems of polynomial eqns, and solve Smale's 17th problem.

# Restrictions

- $w(f)$ is defined as the smallest $n$ such that $\exists$ homog. lin. $\ell_{i,j}^{(k)}$ such that

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**BUT $\mathrm{stab}(\Gamma_{n,d})$ is not reductive!**

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$\Xi_{n,d}$ has connected reductive stabilizer and is characterized by its stabilizer (if one defines $\leq$ correctly).

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# Kronecker product and the matrix multiplication exponent

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- Used today for the Coppersmith–Winograd tensor to obtain the fastest matrix multiplication algorithms.
- Many tasks in computational linear algebra run in roughly the same time:
  - matrix multiplication
  - solving systems of linear equations
  - determinant
  - matrix inverse
  - $LU$ decomposition
  - etc.

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Our goal: Define $\Xi_{n,d} \boxtimes \Xi_{n,d}$

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## Definition in char 0

Let $h, f \in \mathbb{C}[x_1, \dots, x_N]_d$. $\quad h \boxtimes f \;:=\; \frac{1}{d!}\mathrm{resti}(\mathrm{polar}(h) \boxtimes \mathrm{polar}(f))$

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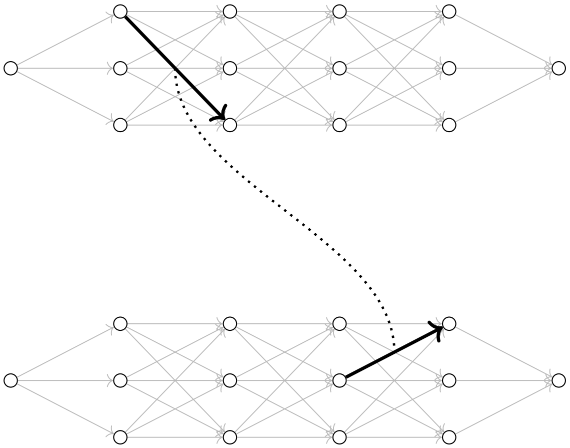
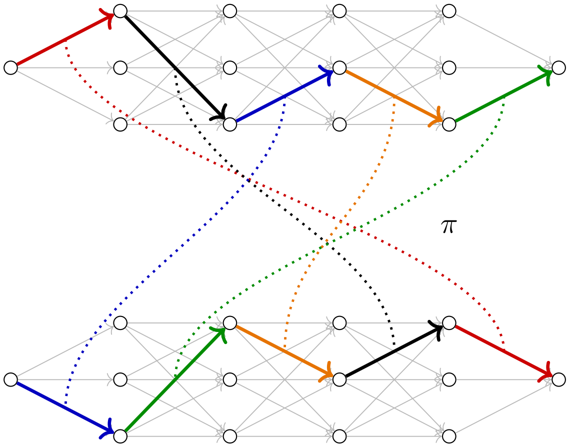
- Corollary: $\mathrm{mon}_{2d} \boxtimes Q_{2d} \;\le\; \mathrm{mon}_{2d} \boxtimes \det_{2d}$, strengthening Gurvits's proof
- Analogous Corollary: $\mathrm{mon}_{2d} \boxtimes Q_{2d} \;\le\; \mathrm{mon}_{2d} \boxtimes \mathrm{per}_{2d} = \mathrm{mon}_{2d}^{\boxtimes 3}$, the hyperpermanent

# Aside: The Boolean decision version of $\Xi_{n,d} \boxtimes \Xi_{n,d}$

Given a set of pairs of edges, find a size $d$ subset that forms two source-sink paths.



Every solution gives a permutation $\pi$ between the top and bottom edge layers:



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Preliminary calculations in the paper: polystability, stabilizers, representation theoretic decompositions

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- Ahmed Shaaban 2026 (using GPT 5.6) proved: There is a commutative ring where determinantal complexity gives a different exponent.

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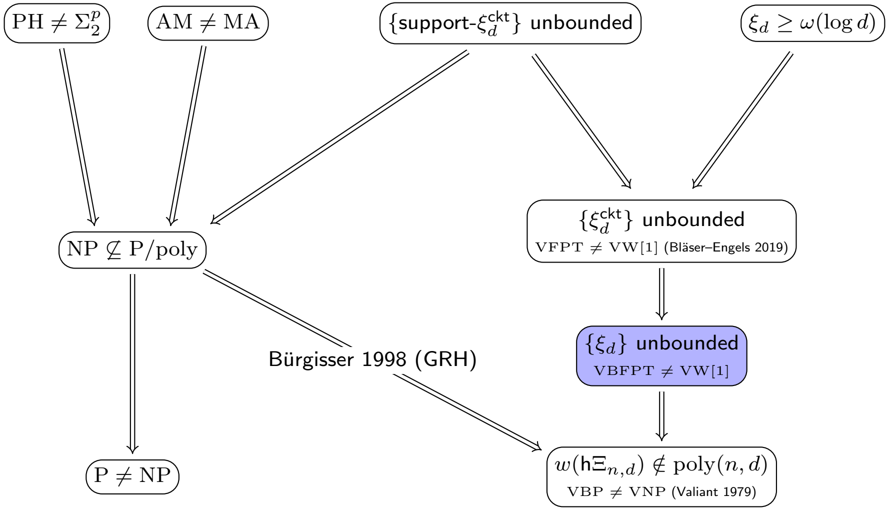

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- Instead of polynomials, study **double-indexed** ($n$ and $d$) **matrices** of polynomials.
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- Both properties at the same time: $\Xi_{n,d}$ is characterized by reductive stabilizer, and the tensor variant is Zariski-closed.
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Thank you for your attention!