# Derandomizing PIT: A Survey of Results and Techniques

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# Outline

- The PIT problem
- PIT and circuit lower bounds
- PIT for constant depth circuits
- PIT for constant read circuits
- PIT for orbits of circuit classes

# Polynomial Identity Testing (PIT)

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The coefficients of all monomials are $0$.
Denoted $f \equiv 0$.

Not the same as $f(a_1, \dots, a_n) = 0$
$\forall a_1, \dots, a_n \in \mathbb{F}$. Eg. $x^2 - x$ over $\mathbb{F}_2$.

# Polynomial Identity Testing (PIT)

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List of coefficients: Problem trivial

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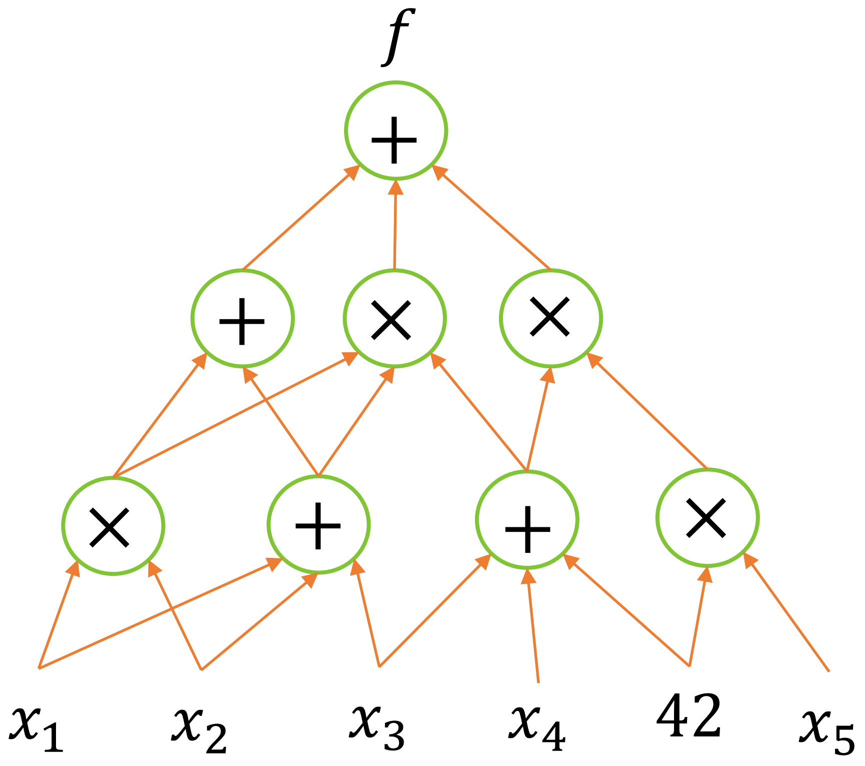
Arithmetic circuit



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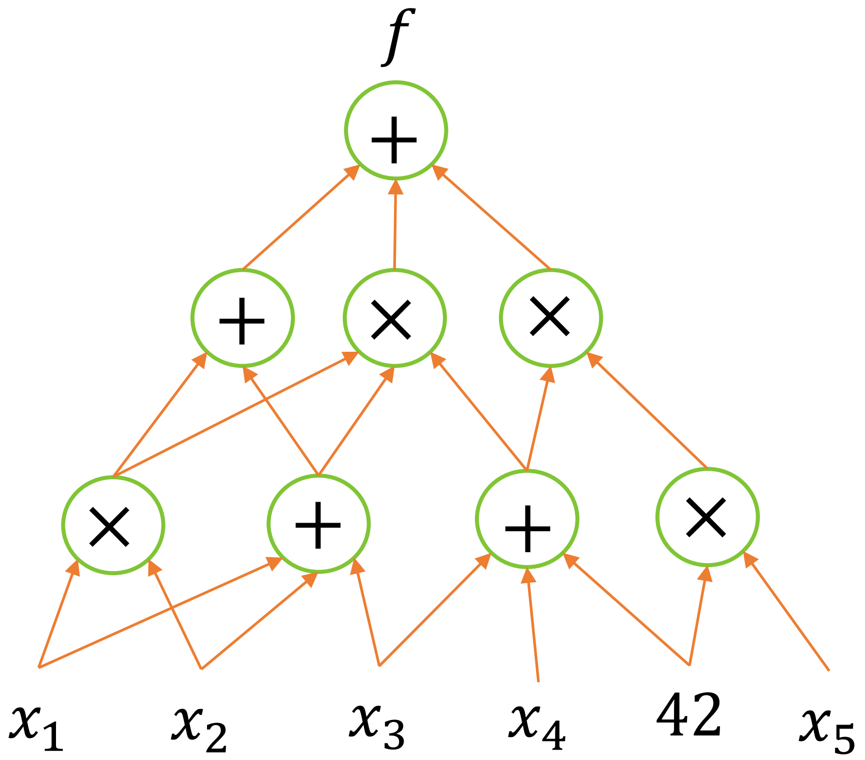
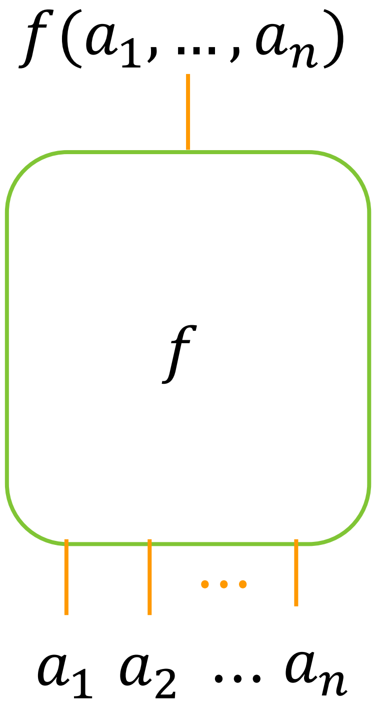



Black box access



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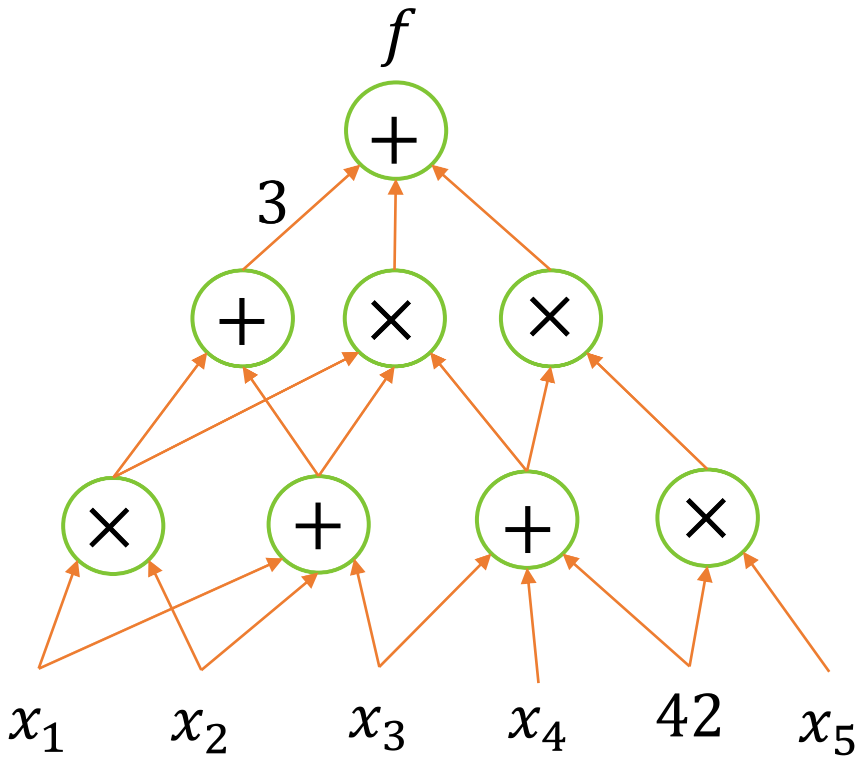
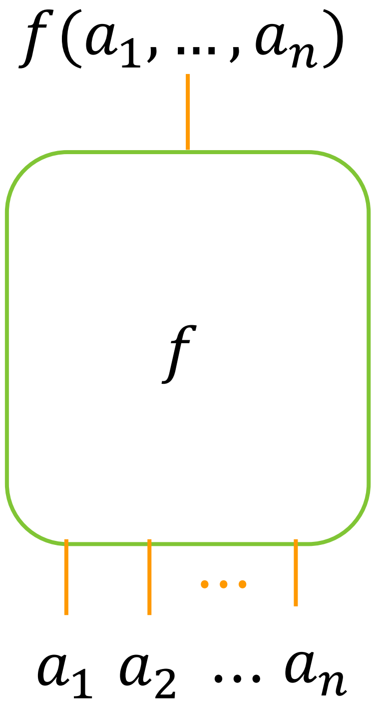
White box PIT




Black box PIT

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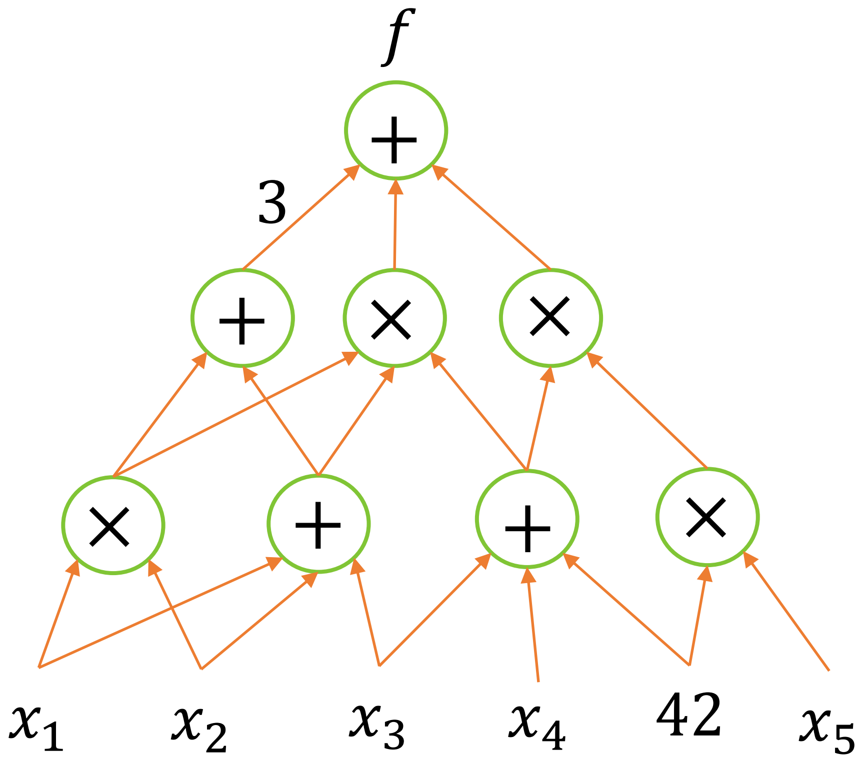
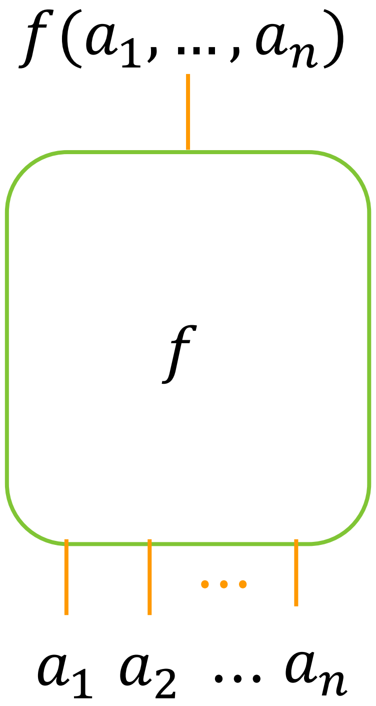



Hitting sets




along with $n, d, s$.

# Efficient randomised algorithm

- **Schwartz-Zippel Lemma** [DL78, Zip79, Sch80]**:** Let $f \in \mathbb{F}[x_1, \dots, x_n]$ be a non-zero, degree $d$ polynomial. Then, for any $S \subseteq \mathbb{F}$ and $a_1, \dots, a_n \in_R S$,
$$\Pr[f(a_1, \dots, a_n) \neq 0] \geq 1 - \frac{d}{|S|}.$$

- Gives a $\mathrm{poly}(n, d)$ randomised algorithm for PIT: Pick $a_1, \dots, a_n$ uniformly at random from a large enough subset of $\mathbb{F}$ and check whether $f(a_1, \dots, a_n)$ is $0$.

- **Goal:** Obtain an <u>efficient, deterministic</u> algorithm for PIT.


DL78: DeMillo-Lipton, Information Processing Letters, 78.
Zip79: Zippel, EUROSAM, 79.
Sch80: Schwartz, JACM, 80.

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Running time $= \mathrm{poly}(n, d, s)$.

# Connections to other problems

- **Primality testing:** The AKS primality test was obtained by derandomizing an instance of PIT over a ring.

- **Perfect matchings:** The best known randomised parallel algorithm for finding perfect matchings in graphs uses PIT [MVV87]. Derandomizing PIT will give a deterministic parallel algorithm to find perfect matchings in graphs.

- **Polynomial factoring:** A deterministic algorithm for PIT would yield a deterministic algorithm for polynomial factorisation [KSS15].

MVV87: Mulmuley-Vazirani-Vazirani, STOC, 87.
KSS15: Kopparty-Saraf-Shpilka, CCC, 14.

# PIT and circuit lower bounds

- **Theorem** [KI03]: If there is a sub-exponential time algorithm for PIT, then either:

  1. There is a function in $\mathsf{NEXP}$ that can not be computed by polynomial sized Boolean circuits or
  2. the permanent polynomial can not be computed by polynomial sized arithmetic circuits.

KI03: Kabanets-Impagliazzo, STOC, 03.

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$$Perm\begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{bmatrix} := \sum_{\sigma\in S_n}\prod_{i\in[n]} x_{i,\sigma(i)}$$

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- The result applies to both the white box and the black box setting.

# PIT and circuit lower bounds

- **Theorem** [HS80, Agr05]: Let $T : \mathbb{N} \to \mathbb{N}$ be an increasing function. Suppose there is an algorithm which runs in time $T(s)$ and solves the black box version of PIT for size $s$ circuits. Then there exists an $n$ variate polynomial whose coefficients can be computed in time $2^{O(n)}$ that requires arithmetic circuits of size at least $T^{-1}\left(2^{O(n)}\right)$.

HS80: Heintz-Schnorr, STOC, 80.
Agr05: Agrawal, FSTTCS, 05.

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- Polynomial time black box PIT $\Longrightarrow$ exponential arithmetic circuit lower bound.

- Quasi-polynomial time black box PIT $\Longrightarrow$ arithmetic circuit lower bound of the form $2^{n^{\epsilon}}$.

# PIT and circuit lower bounds

- **Theorem** [KI03]: If there is an $n$ variate, multilinear polynomial that requires arithmetic circuits of size $2^{\Omega(n)}$ (resp. $n^{\omega(1)}$), then there is a $2^{\mathrm{polylog}(n)}$ (resp. sub-exponential) time black box PIT algorithm for $\mathrm{poly}(n)$ sized arithmetic circuits computing $n$ variate polynomials of $\mathrm{poly}(n)$ degree.

- Thus, derandomizing PIT and proving arithmetic circuit lower bounds are <u>two sides of the same coin</u>.

KI03: Kabanets-Impagliazzo, STOC, 03.

# PIT for special circuit classes

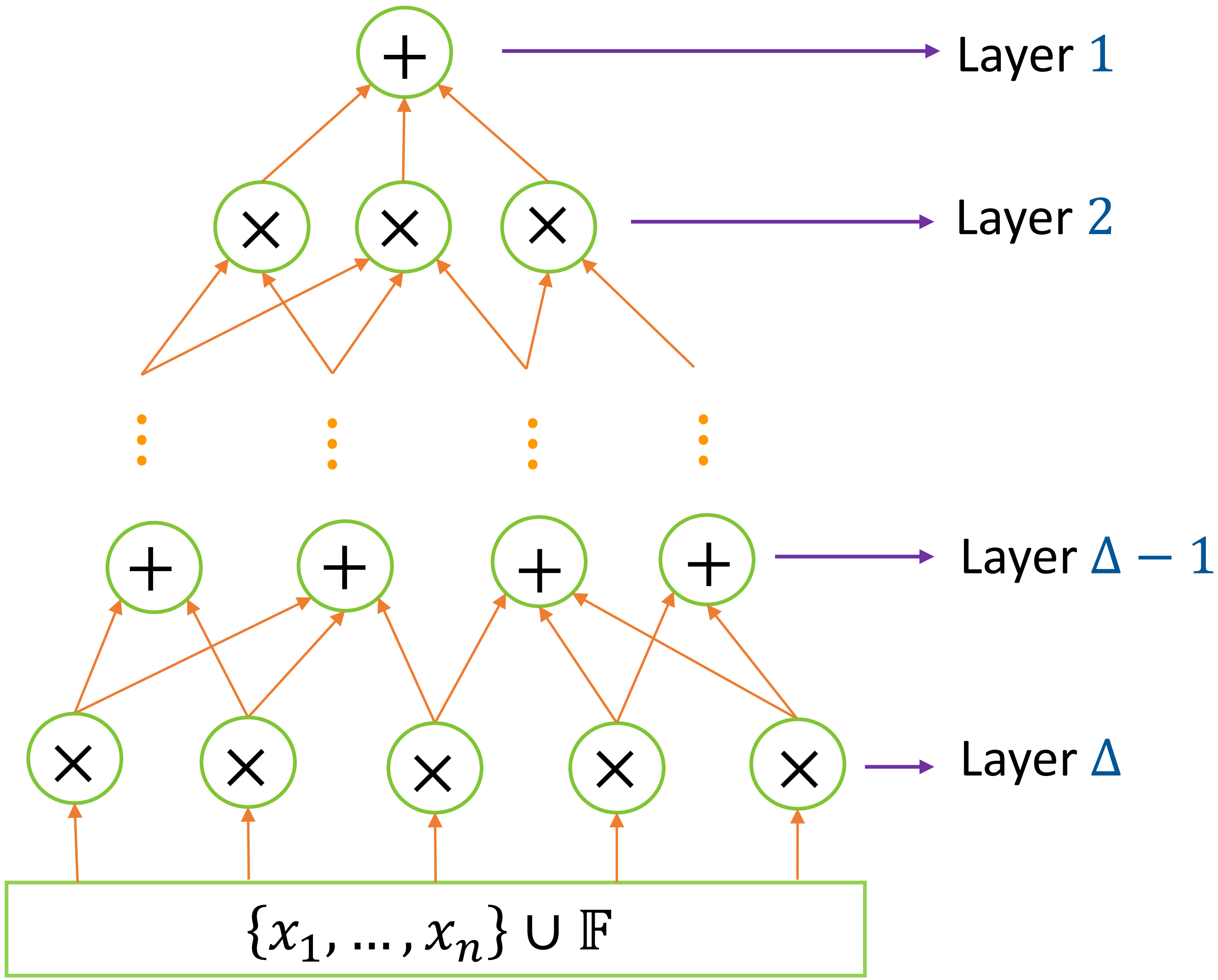
- Since proving arithmetic circuit lower bounds seems to be difficult, we can expect derandomizing PIT to be a challenging problem.

- So the focus has been on derandomizing PIT for special classes of circuits.

- Some restrictions that have been imposed are:
  - Restricting the depth of the circuit,
  - Restricting the number of times the circuit can read a variable,
  - Restricting the fan-in of the gates in the circuit,
  - Combinations of the above three, etc.

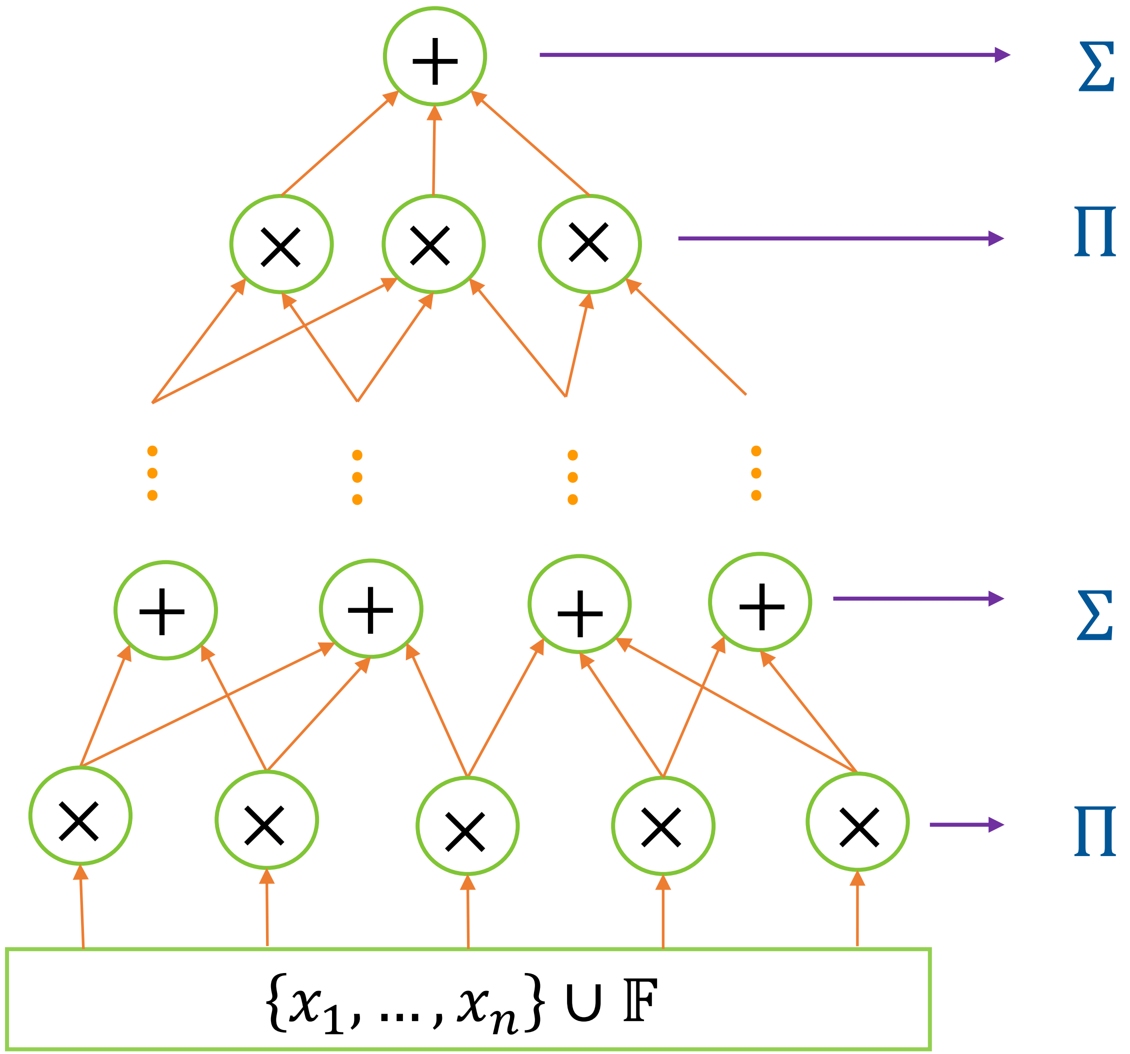
# PIT for Constant Depth Circuits

# Constant depth circuits



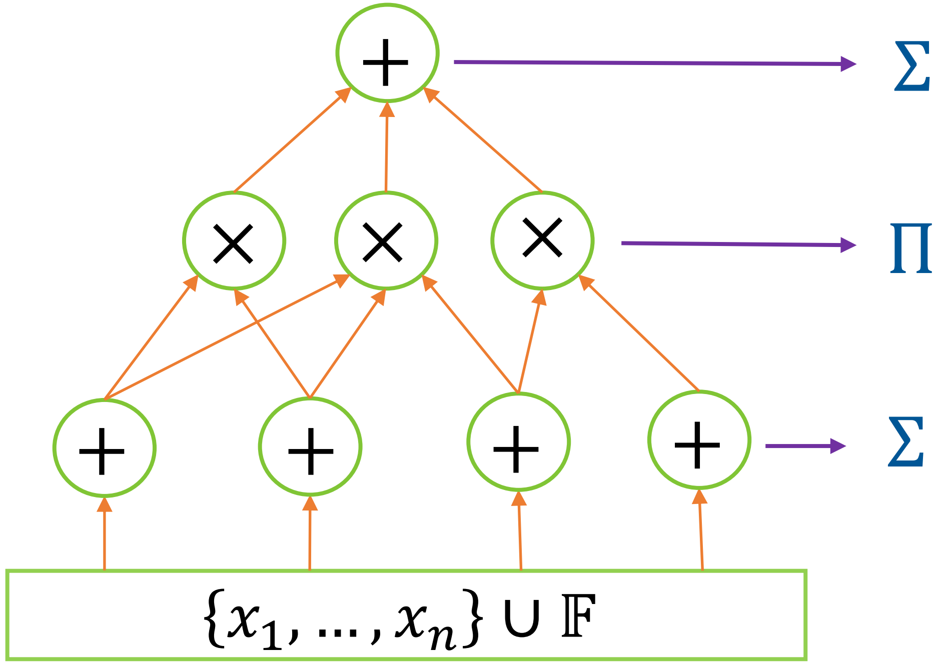

- Alternating layers/levels of $+$ and $\times$ gates with unbounded fan-in.

# Constant depth circuits




- Alternating layers/levels of $+$ and $\times$ gates with unbounded fan-in.
- Every layer of $+$ gates is denoted by $\Sigma$. Every layer of $\times$ gates is denoted by $\Pi$.
- Every depth $\Delta$ cirucit can be denoted by a string of length $\Delta$ consisting of alternating $\Sigma$s and $\Pi$s.

# Constant depth circuits




A $\Sigma\Pi\Sigma$ circuit

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# $\Sigma\Pi$ circuits

- A $\Sigma\Pi$ circuit (aka a sparse polynomial) computes an $\mathbb{F}$-linear combination of monomials and is thus a universal model of computation.

- **White box PIT:** Trivial.

- **Black box PIT** [KS01]**:** There is a $\mathrm{poly}(n, d, s)$ time black box PIT algorithm for the class of $n$ variate, degree $d$, $s$ sparse polynomials over fields of size $\mathrm{poly}(n, d, s)$.

KS01: Klivans-Spielman, STOC, 01.

# ΣΠ circuits – black box PIT

- Let $f = \sum_{i\in[s]} c_i \cdot x_1^{d_{i,1}} \cdots x_n^{d_{i,n}}$ be a non-zero, degree $d$, $s$ sparse polynomial.
- Map $x_i \mapsto x^{t^{i-1} \bmod q}, \forall i \in [n]$, where $q$ is a prime number $> s^2nd$. Thus the monomial $x_1^{d_{i,1}} \cdots x_n^{d_{i,n}}$ maps to $x^{d_{i,n}(t^{n-1} \bmod q)+d_{i,n-1}\,(t^{n-2} \bmod q)+\cdots+d_{i,1}}$.
- Let $p_i(t) = d_{i,n}(t^{n-1} \bmod q) + d_{i,n-1}\,(t^{n-2} \bmod q) + \cdots + d_{i,1}$. We find an $\alpha \in \mathbb{N}$ s.t. $\forall i \neq j, p_i(\alpha) \neq p_j(\alpha) \bmod q$. Then $\forall i \neq j, p_i(\alpha) \neq p_j(\alpha)$.

# $\Sigma\Pi$ circuits – black box PIT

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Any $\alpha$ which is not a root of $\prod_{i\neq j}\left(p_i(t) - p_j(t)\right) \bmod q$ over $\mathbb{F}_q$ will work. As $q > s^2n$ such an $\alpha$ exists.

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- Now $f\left(x, x^{\alpha \bmod q}, \ldots, x^{\alpha^{n-1} \bmod q}\right)$ is a non-zero, univariate polynomial of degree $\leq dq$. Thus, by trying out $\leq dq + 1$ many values for $x$, we find a $\beta \in \mathbb{F}$ s.t. $f\left(\beta, \beta^{\alpha \bmod q}, \ldots, \beta^{\alpha^{n-1} \bmod q}\right) \neq 0$.

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Such a $\beta$ will exist as $|\mathbb{F}| =$ $\mathrm{poly}(n, d, s)$.

# $\Sigma\Pi$ circuits – black box PIT

- **Running time of the algorithm:** The algorithm finds $q$, tries at most $s^2 n + 1$ many values of $\alpha$ and for each value of $\alpha$, tries at most $dq + 1$ many values of $\beta$.

- A prime $s^2 nd < q \leq 2s^2 nd$ exists and can be found in $\mathrm{poly}(n, d, s)$ time. Time required to try various values of $\alpha$ and $\beta$ is $\leq (s^2 n + 1)(dq + 1) = \mathrm{poly}(n, d, s)$. Total time = $\mathrm{poly}(n, d, s)$.

# $\Sigma\Pi\Sigma$ circuits

- **Theorem** [VSBR83, AV08, Koi12, GKKS13, Tav13]**:** If $f$ is an $n$ variate, degree $\mathrm{poly}(n)$ polynomial computed by a $\mathrm{poly}(n)$ size circuit, then it can also be computed by a $\Sigma\Pi\Sigma$ circuit of size $n^{O(\sqrt{n})}$.

- <u>Polynomial time</u> PIT for $\Sigma\Pi\Sigma$ circuits $\Longrightarrow$ <u>sub-exponential</u> PIT for $\mathrm{poly}(n)$ size circuits computing $\mathrm{poly}(n)$ degree polynomials. PIT for $\Sigma\Pi\Sigma$ circuits is as challenging as PIT for general circuits.

- Researchers have studied restricted classes of $\Sigma\Pi\Sigma$ circuits.


VSBR83: Valiant-Skyum-Berkowitz-Rackoff, SICOMP, 83.
AV08: Agrawal-Vinay, FOCS, 08.
Koi12: Koiran, Theor. Comput. Sci., 12.
GKKS13: Gupta-Kamath-Kayal-Saptharishi, FOCS, 13.
Tav13: Tavenas, MFCS 13.

# $\Sigma^k\Pi^d\Sigma$ circuits

- A $\Sigma^k\Pi^d\Sigma$ circuit is a $\Sigma\Pi\Sigma$ circuit where the fan-in of the top $+$ gate is at most $k$ and the fan-in of all product gates in the second level is at most $d$. Think of $k$ as a constant.

- Both white box and black box PIT for $\Sigma^k\Pi^d\Sigma$ circuits have been studied extensively.

# PIT for $\Sigma^k \Pi^d \Sigma$ circuits

| Paper | Version | Result |
|---|---|---|
| DS05 | White box | $\mathrm{poly}(n, d^{O(k^2 \log^{k-2} d)})$ |
| KS06 | White box | $\mathrm{poly}(n, d^{O(k)})$ |
| KS08 | Black box | $\mathrm{poly}(n, d^{O(k^2 \log^{k-2} d)})$ |
| SS09 | Black box | $\mathrm{poly}(n, d^{O(k^3 \log d)})$ |
| KS09 | Black box | $\mathrm{poly}(n, d^{O(k^k)})$ over $\mathbb{R}$ |
| SS10 | Black box | $\mathrm{poly}(n, d^{O(k^2)})$ over $\mathbb{R}$<br>$\mathrm{poly}(n, d^{O(k^2 \log d)})$ over any $\mathbb{F}$ |
| SS11 | Black box | $\mathrm{poly}(n, d^{O(k)})$ |


DS05: Dvir-Shpilka, STOC, 05.
KS06: Kayal-Saxena, CCC, 06.
KS08: Karnin-Shpilka, CCC, 08.
SS09: Saxena-Seshadhri, CCC, 09.
KS09: Kayal-Saraf, FOCS, 09.
SS10: Saxena-Seshadhri, FOCS, 10.
SS11: Saxena-Seshadhri, STOC, 11.

# An approach for $\Sigma^k\Pi^d\Sigma$ black box PIT

- Let $f = T_1 + \cdots + T_k$, $T_i = \ell_{i,1} \cdots \ell_{i,d_i}$, where $\ell_{i,j}$ are linear polynomials, be a $\Sigma^k\Pi^d\Sigma$ circuit computing an $n$ variate polynomial.

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- A lot of black box PIT algorithms for $\Sigma^k\Pi^d\Sigma$ circuits use the rank bound idea.

- $\mathrm{rank}(f) \coloneqq \dim \mathrm{span}\{\ell_{1,1}, \ldots, \ell_{k,d}\}$.

# rank and $\Sigma^k\Pi^d\Sigma$ PIT

- Suppose $\mathrm{rank}(f) = r$. Let $\{\ell_{i_1,j_1}, \dots, \ell_{i_r,j_r}\}$ be a basis of $\mathrm{span}\{\ell_{1,1}, \dots, \ell_{k,d}\}$.

- **Rank extractors:** Let $V$ be an unknown but fixed space of linear functions from $\mathbb{F}^n$ to $\mathbb{F}$ of dimension at most $r$. [GR05] showed that a linear transformation $T: \mathbb{F}^r \to \mathbb{F}^n$ s.t. $\dim V \circ T = \dim V$ can be constructed in $\mathrm{poly}(n, r)$ time provided that $|\mathbb{F}| = \mathrm{poly}(n, r)$.

- $V := \mathrm{span}\{\ell_{i_1,j_1}, \dots, \ell_{i_r,j_r}\}$. It is not to difficult to show that $f \equiv 0 \iff f \circ T \equiv 0$.

- $f \circ T$ is an $r$ variate polynomial. If $r$ is “small” we can find a non-root of $f \circ T$ by brute force search.

GR05: Gabizon-Raz, FOCS, 05.

# rank and $\Sigma^k\Pi^d\Sigma$ PIT

- We can not expect the rank of an arbitrary $\Sigma^k\Pi^d\Sigma$ circuit to be small.

- However, it turns out that a rank bound for simple and minimal $\Sigma^k\Pi^d\Sigma$ circuits computing the $0$ polynomial suffices.

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$f = T_1 + \cdots + T_k$ is simple if there is no linear form that divides all of $T_1, \ldots, T_k$.

# rank and $\Sigma^k\Pi^d\Sigma$ PIT

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$f = T_1 + \cdots + T_k$ is minimal if $\forall S \subseteq [k], \sum_{i\in S} T_i \not\equiv 0$.

# rank and $\Sigma^k\Pi^d\Sigma$ PIT

- We can not expect the rank of an arbitrary $\Sigma^k\Pi^d\Sigma$ circuit to be small.
- However, it turns out that a rank bound for simple and minimal $\Sigma^k\Pi^d\Sigma$ circuits computing the $0$ polynomial suffices.
- **Theorem** [KS06]**:** Suppose that the rank of all $n$ variate simple and minimal $\Sigma^k\Pi^d\Sigma$ circuits computing the $0$ polynomial is at most $R(k,d)$. Then, there is an $\mathrm{poly}(n, 2^k, d^{R(k,d)})$ time black box PIT algorithm for $\Sigma^k\Pi^d\Sigma$ circuits.
- The proof of the above theorem crucially uses the rank extractors from [GR05].

KS06: Karnin-Shpilka, CCC, 06.
GR05: Gabizon-Raz, FOCS, 05.

# rank and $\Sigma^k\Pi^d\Sigma$ PIT

- How can we show that the rank of every simple and minimal $\Sigma^k\Pi^d\Sigma$ circuit computing the $0$ polynomial is "small"?

- One way is to use Sylvester-Gallai type theorems.

# A detour: Sylvester-Gallai theorem

- **Sylvester-Gallai Theorem:** Let $S \subseteq \mathbb{R}^2$ be a finite set. If $\forall \boldsymbol{a}, \boldsymbol{b} \in S, \exists \boldsymbol{c} \in S$, s.t. the line passing through $\boldsymbol{a}$ and $\boldsymbol{b}$ also contains $\boldsymbol{c}$, then all points in $S$ are collinear.

- **Edelstein-Kelly Theorem:** Let $R, G, B \subseteq \mathbb{R}^2$ be disjoint, finite sets of the same size. If for every pair of points $\boldsymbol{a}, \boldsymbol{b}$ from two distinct sets, there exists $\boldsymbol{c}$ in the third set, s.t. the line passing through $\boldsymbol{a}$ and $\boldsymbol{b}$ also contains $\boldsymbol{c}$, then all points in $R \cup G \cup B$ are collinear.

# rank and $\Sigma^k\Pi^d\Sigma$ PIT

- How can we show that the rank of every simple and minimal $\Sigma^k\Pi^d\Sigma$ circuit computing the $0$ polynomial is “small”?

- Let $f = T_1 + T_2 + T_3$ be a simple and minimal $\Sigma^k\Pi^d\Sigma$ circuit computing the $0$ polynomial. Let $T_i = \ell_{i,1} \cdots \ell_{i,d}$ and $S_i = \{\ell_{i,1}, \ldots, \ell_{i,d}\}$. Since $f$ is simple, the $S_i$ are disjoint. Now, $0 \equiv f \bmod \ell_{1,1} = (T_2 + T_3) \bmod \ell_{1,1} \implies \forall \ell_{2,j}, \exists\, \ell_{3,j'}$ s.t. $\ell_{3,j'} = \ell_{2,j} \bmod \ell_{1,1}$. I.e. $\ell_{3,j'} \in \mathrm{span}\{\ell_{2,j}, \ell_{1,1}\}$. Thus, $S_1, S_2, S_3$ have a structure like the one found in the hypothesis of the Edelstein-Kelly Theorem. Perhaps this can be used to bound the rank.

- Sylvester-Gallai type theorems were used to bound rank in [KS09, SS10].

KS09: Kayal-Saraf, FOCS, 09.
SS10: Saxena-Seshadhri, FOCS, 10.

# $\Sigma^k\Pi^d\Sigma$ black box PIT

- **Summary:**

1. Rank bound on simple, minimal $\Sigma^k\Pi^d\Sigma$ circuits computing the $0$ polynomial + Rank extractors imply black box PIT for $\Sigma^k\Pi^d\Sigma$ circuits.

2. Sylvester-Gallai type theorems can be used to prove that the rank of simple, minimal $\Sigma^k\Pi^d\Sigma$ circuits computing the $0$ polynomial is "small".

# $\Sigma \wedge \Sigma$ circuits

- $\Sigma \wedge \Sigma$ circuits are a natural sub-class of $\Sigma\Pi\Sigma$ circuits.

- A $\Sigma \wedge \Sigma$ circuit looks like $\sum_{i\in[k]} \ell_i^d$. I.e. all the inputs of a $\times$ gate in the second level are the same.

- [Sax08, FS13] showed that $\Sigma \wedge \Sigma$ circuits are a sub-class of Read-once Oblivious Algebraic Branching Programs (ROABPs).

- This observation yields polynomial time white box and quasi-polynomial time black box PIT algorithms for this model.

Sax08: Saxena, ICALP, 08.
FS13: Forbes-Shpilka, FOCS, 13.

# Depth 4 circuits

- **Theorem** [VSBR83, AV08, Koi12, GKKS13, Tav13]**:** If $f$ is an $n$ variate, degree $\mathrm{poly}(n)$ polynomial computed by a $\mathrm{poly}(n)$ size circuit, then it can also be computed by a $\Sigma\Pi\Sigma\Pi$ circuit of size $n^{O(\sqrt{n})}$.

VSBR83: Valiant-Skyum-Berkowitz-Rackoff, SIAM J. Comput., 83.
AV08: Agrawal-Vinay, FOCS, 08.
Koi12: Koiran, Theor. Comput. Sci., 12.
GKKS13: Gupta-Kamath-Kayal-Saptharishi, FOCS, 13.
Tav13: Tavenas, MFCS 13.

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In fact, by circuits where $\times$ gates have fan-in $O(\sqrt{n})$.


VSBR83: Valiant-Skyum-Berkowitz-Rackoff, SIAM J. Comput., 83.
AV08: Agrawal-Vinay, FOCS, 08.
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# Depth 4 circuits

- **Theorem** [VSBR83, AV08, Koi12, GKKS13, Tav13]**:** If $f$ is an $n$ variate, degree $\text{poly}(n)$ polynomial computed by a $\text{poly}(n)$ size circuit, then it can also be computed by a $\Sigma\Pi\Sigma\Pi$ circuit of size $n^{O(\sqrt{n})}$.

- Polynomial time PIT for $\Sigma\Pi\Sigma\Pi$ circuits $\implies$ sub-exponential PIT for $\text{poly}(n)$ size circuits computing $\text{poly}(n)$ degree polynomials.

- A natural sub-class to study is $\Sigma^k\Pi\Sigma\Pi^\delta$ circuits.

# PIT for $\Sigma^k\Pi\Sigma\Pi^\delta$ circuits?

- One natural approach is to generalise the notion of rank, rank extractors, and Sylvester-Gallai type theorems used for $\Sigma^k\Pi^d\Sigma$ circuits to appropriate notions for $\Sigma^k\Pi\Sigma\Pi^\delta$ circuits. This was done in [BMS11, Gup14].

- [BMS11] replaces rank by transcendence degree.

BMS11: Beecken-Mittmann-Saxena, ICALP, 11.
Gup14: Gupta, ECCC, 14.

# A detour: algebraic independence

- $f_1, \dots, f_m \in \mathbb{F}[x_1, \dots, x_n]$ are said to be algebraically independent if there does not exist any non-zero $P \in \mathbb{F}[y_1, \dots, y_m]$ s.t. $P(f_1, \dots, f_m) \equiv 0$.

- $\mathbb{F}[x_1, \dots, x_n]$ forms a matroid under algebraic independence.

- **Transcendence degree:** For any $S \subseteq \mathbb{F}[x_1, \dots, x_n]$, the transcendence degree of $S$, denoted by $\mathrm{tr} - \deg(S)$, is the size of the maximum cardinality set of algebraically independent polynomials in $S$. It can be shown that $\mathrm{tr} - \deg(S) \leq n$.

# PIT for $\Sigma^k\Pi\Sigma\Pi^\delta$ circuits?

- One natural approach is to generalise the notion of rank, rank extractors, and Sylvester-Gallai type theorems used for $\Sigma^k\Pi^d\Sigma$ circuits to appropriate notions for $\Sigma^k\Pi\Sigma\Pi^\delta$ circuits. This was done in [BMS11, Gup14].

- [BMS11] replaces rank by transcendence degree. Let $f = \sum_{i\in[k]}\prod_{j\in[s]} f_{i,j}$ be a $\Sigma^k\Pi\Sigma\Pi^\delta$ circuit. Then,

$$\mathrm{rank}(f) \coloneqq \mathrm{tr}-\deg\left\{f_{i,j}\right\}_{i,j}.$$

# PIT for $\Sigma^k \Pi\Sigma\Pi^\delta$ circuits?

- [BMS11] replaces rank extractors by faithful homomorphisms.

BMS11: Beecken-Mittmann-Saxena, ICALP, 11.

# PIT for $\Sigma^k \Pi \Sigma \Pi^\delta$ circuits?

- [BMS11] replaces rank extractors by faithful homomorphisms.

$\phi: \mathbb{F}[x_1, \dots, x_n] \to \mathbb{F}[y_1, \dots, y_m]$ s.t.
$\forall p, q \in \mathbb{F}[x_1, \dots, x_n]$,
$\phi(p + q) = \phi(p) + \phi(q)$ and
$\phi(pq) = \phi(p)\phi(q)$.

# PIT for $\Sigma^k\Pi\Sigma\Pi^\delta$ circuits?

- [BMS11] replaces rank extractors by faithful homomorphisms.

$\phi: \mathbb{F}[x_1, \ldots, x_n] \rightarrow \mathbb{F}[y_1, \ldots, y_m]$ is said to be faithful to $\{f_1, \ldots, f_s\}$ if

$$\mathrm{tr}-\deg\{f_1, \ldots, f_s\} = \mathrm{tr}-\deg\{\phi(f_1), \ldots, \phi(f_s)\}.$$

# PIT for $\Sigma^k\Pi\Sigma\Pi^\delta$ circuits?

- [BMS11] replaces rank extractors by faithful homomorphisms.

- **Theorem** [BMS11]**:** If the rank of every $n$ variate simple and minimal size $s$ $\Sigma^k\Pi\Sigma\Pi^\delta$ circuit computing the $0$ polynomial is at most $r$, then there is a black box PIT algorithm for size-$s$ $\Sigma^k\Pi\Sigma\Pi^\delta$ circuits that runs in time $\mathrm{poly}(n, r, \delta, s)^{\delta^2 kr}$.

# PIT for $\Sigma^k\Pi\Sigma\Pi^\delta$ circuits?

- To bound the rank of simple, minimal $\Sigma^k\Pi\Sigma\Pi^\delta$ circuit computing the $0$ polynomial, [Gup14] proposed a Sylvester-Gallai type conjecture for $\Sigma^k\Pi\Sigma\Pi^\delta$ circuits.

- [Shp19, PS20, PS21] proved Gupta's conjecture for $\Sigma^3\Pi\Sigma\Pi^2$ circuits thereby obtaining a black box, $\mathrm{poly}(n,d)$ PIT algorithm for $\Sigma^3\Pi\Sigma\Pi^2$ circuits.


Gup14: Gupta, ECCC, 14.
Shp19: Shpilka, FOCS, 19.
PS20: Peleg-Shpilka, CCC, 20.
PS21: Peleg-Shpilka, STOC, 21.

# PIT for depth 4 circuits

| Model | Paper | Version | Result |
|---|---|---|---|
| $\Sigma^k \Pi^\delta \Sigma \wedge$ | Sax08 | White box | $\mathrm{poly}(n, k, s^{O(\delta)})$ |
| Multilinear $\Sigma^k \Pi \Sigma \Pi$ | SV11, ASSS12 | Black box | $\mathrm{poly}(n^{O(k^2)})$ |
| $\Sigma^2 \Pi \Sigma \Pi^\delta$ | BMS11 | Black box | $\mathrm{poly}(n, \delta, s)^{\delta^2}$ |
| $\Sigma \wedge \Sigma \Pi^\delta$ | For15 | Black box | $s^{O(\delta \log s)}$ |
| $\Sigma^3 \Pi \Sigma \Pi^2$ | PS21 | Black box | $\mathrm{poly}(n, d)$ |
| $\Sigma^k \Pi \Sigma \wedge$ | DDS20 | White box | $s^{O(k\, 7^k)}$ |
| $\Sigma^k \Pi \Sigma \wedge$ | DDS20 | Black box | $s^{O(k \log \log s)}$ |
| $\Sigma^k \Pi \Sigma \Pi^\delta$ | DDS20 | Black box | $s^{O(\delta^2 k \log s)}$ |
| $\overline{\Sigma^k \Pi \Sigma \wedge}$ | DDS21 | Black box | $s^{O(k\, 7^k \log \log s)}$ |
| $\overline{\Sigma^k \Pi \Sigma \Pi^\delta}$ | DDS21 | Black box | $s^{O(\delta^2 k\, 7^k \log s)}$ |

Sax08: Saxena, ICALP, 08.
SV11: Saraf-Volkovich, STOC, 11.
ASSS12: Agrawal-Saha-Sapthirishi-Saxena, STOC, 12.
BMS11: Beecken-Mittmann-Saxena, ICALP, 11.
For15: Forbes, FOCS, 15.
PS21: Peleg-Shpilka, STOC, 21.
DDS20: Dutta-Dwivedi-Saxena, CCC, 2021.
DDS21: Dutta-Dwivedi-Saxena, FOCS, 2021.

# PIT for low depth circuits

- In a breakthrough paper [LST21], Limaye, Srinivasan, and Tavenas proved super-polynomial lower bounds for low depth circuits.

- [DSY08, CKS19] showed that super-polynomial lower bounds for low depth circuits imply sub-exponential PIT for such circuits.

- Thus, [LST21] yields a $\left(n \cdot s^{\Delta+1}\right)^{n^{\epsilon}}, \epsilon > 0$ , time PIT for depth $\Delta = o(\log \log \log n)$ circuits provided that $s = \mathrm{poly}(n)$.

LST21: Limaye-Srinivasan-Tavenas, FOCS 21.
DSY08: Dvir-Shpilka-Yehuayoff, STOC, 08.
CKS19: Chou-Kumar-Solomon, CCC, 18.

# PIT for Constant Read Circuits

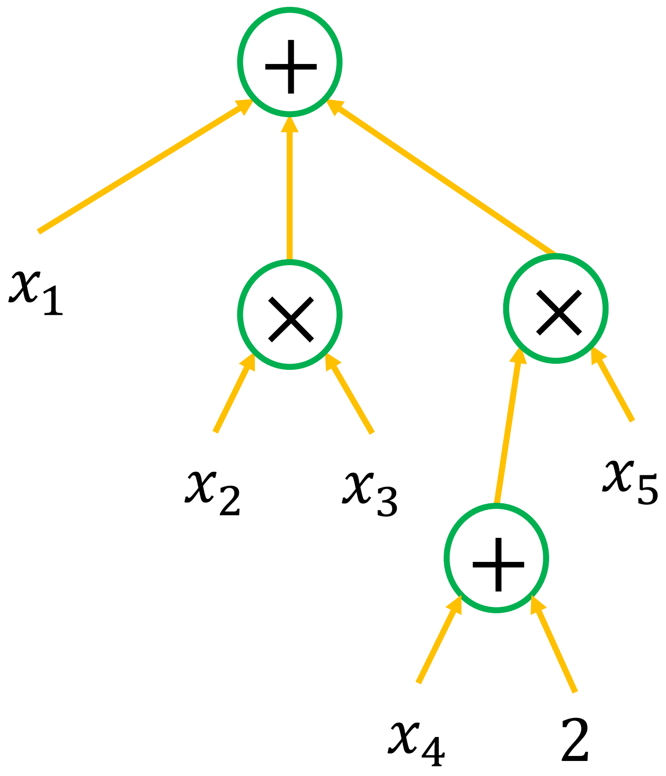
# Read once formulas

- **Arithmetic Formulas:** Arithmetic circuits whose underlying graph is a tree.

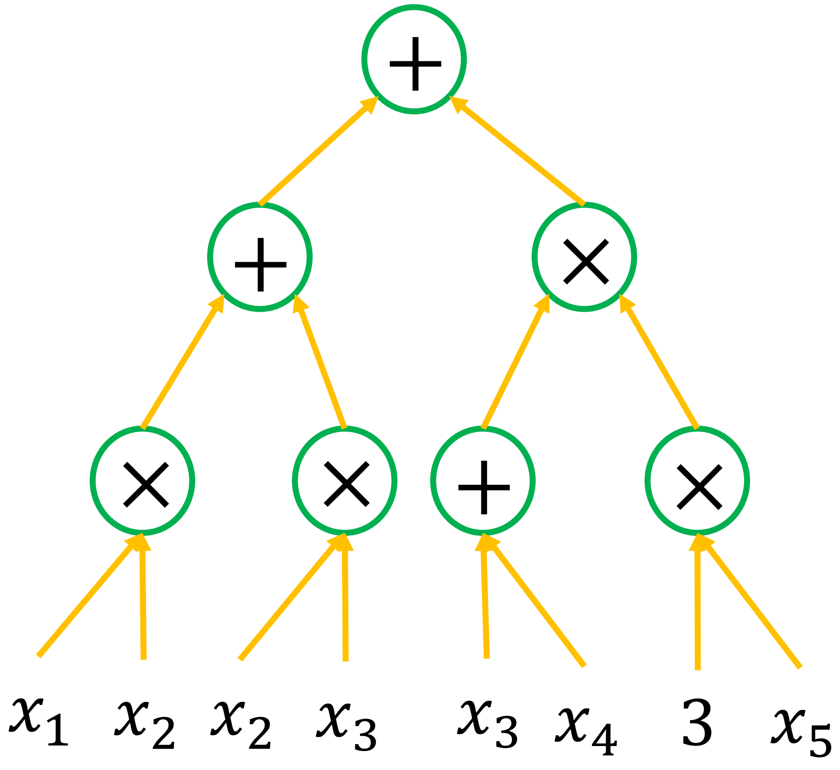
# Read once formulas

- **Read Once Formulas (ROFs):** Arithmetic formulas where each variable appears in at most one leaf.

ROF:




Not an ROF:



# Read once formulas

- **Read Once Formulas (ROFs):** Arithmetic formulas where each variable appears in at most one leaf.
- ROFs are a special class of multilinear circuits.
- [SV09] gave an $n^{O(\log n)}$ time black box PIT algorithm for ROFs.
- This was improved to a $\mathrm{poly}(n)$ time algorithm by [MV17].

SV09: Shpilka-Volkovich , APPROX-RANDOM, 09.
MV17: Minahan-Volkovich, CCC, 17.

# Constant read formulas

- **Read $\boldsymbol{k}$ Formulas:** Arithmetic formulas where each variable appears in at most $k$ leaves.

- [SV09] gave an $n^{O(k+\log n)}$ time black box PIT algorithm for sum of $k \leq \frac{n}{3}$ ROFs.

SV09: Shpilka-Volkovich , APPROX-RANDOM, 09.

# Constant read formulas

- **Read $\boldsymbol{k}$ Formulas:** Arithmetic formulas where each variable appears in at most $k$ leaves.

- [AvMV11] gave a $\mathrm{poly}(s, n^{k^{O(k)}})$ time white box and $n^{k^{O(k)}+O(k \log n)}$ time black box PIT algorithm for multilinear read $k$ formulas.

AvMV11: Anderson-van Melkebeek-Volkovich, CCC, 11.

# Constant read formulas

- **Read $k$ Formulas:** Arithmetic formulas where each variable appears in at most $k$ leaves.

- [ASSS12] gave an $s^{k^{O(\Delta\, 2^{\Delta})}}$ time black box PIT algorithm for occur $k$ formulas of depth $\Delta$ using the algebraic independence technique from [BMS11].

ASSS12: Agrawal-Saha-Sapthirishi-Saxena, STOC, 12.
BMS11: Beecken-Mittmann-Saxena, ICALP, 11.

# Constant read formulas

- **Read $\boldsymbol{k}$ Formulas:** Arithmetic formulas where each variable appears in at most $k$ leaves.

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A generalisation of read $k$ formulas. Capture other interesting models like multilinear $\Sigma^k\Pi\Sigma\Pi$ circuits.

# Read-once oblivious algebraic branching programs

- **ROABP:** $f \in \mathbb{F}[x_1, \dots, x_n]$ is said to be computed by a width-$w$ ROABP in order $\pi \in S_n$ if

$$f = [1, \ \dots, \ 1] \begin{bmatrix} & & \\ & M_1(x_{\pi(1)}) & \\ & & \end{bmatrix}_{w \times w} \cdots\cdots \begin{bmatrix} & & \\ & M_n(x_{\pi(n)}) & \\ & & \end{bmatrix}_{w \times w} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

- The classes of $\Sigma \wedge \Sigma$ and $\Sigma \wedge \Sigma \wedge$ circuits are contained in ROABPs.

- [OSV15] obtained a sub-exponential time black box PIT for multilinear depth $3$ and depth $4$ formulas by reducing to black box PIT for ROABPs.

OSV15: Oliveira-Shpilka-Volk, CCC, 15.

# Read-once oblivious algebraic branching programs

- A $\mathrm{poly}(n, d, w)$ white box PIT for ROABPs follows from [RS04].
- [FS13] gave a $\mathrm{poly}(n, d, w)^{O(\log w)}$ time black box PIT for ROABPs with known variable order.
- [FSS14] gave a $\mathrm{poly}(n, d)^{O(\log w)}$ time black box PIT for multilinear and commutative ROABPs.

RS05: Raz-Shpilka, CCC, 04.
FS13: Forbes-Shpilka, FOCS, 13.
FSS14: Forbes-Saptharishi-Shpilka, STOC, 14.

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$f$ is computed by a width $w$ commutative ROABP if it is computed by a width $w$ ROABP in every variable order.


RS05: Raz-Shpilka, CCC, 04.
FS13: Forbes-Shpilka, FOCS, 13.
FSS14: Forbes-Saptharishi-Shpilka, STOC, 14.

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- [AGKS15] gave a $\mathrm{poly}(n, d, w)^{O(\log n)}$ time black box PIT for ROABPs with unknown variable order.
- [GKST15] gave a $\mathrm{poly}(n, d, w)^{O(\log n)}$ time black box PIT and $\mathrm{poly}(n, d, w)$ time white box PIT for sum of constantly many ROABPs.


RS05: Raz-Shpilka, CCC, 04.
FS13: Forbes-Shpilka, FOCS, 13.
FSS14: Forbes-Saptharishi-Shpilka, STOC, 14.
AGKS15: Agrawal-Gurjar-Korwar-Saxena, SICOMP, 15.
GKST15:  Agrawal-Gurjar-Saxena-Thierauf, CCC, 15.

# PIT for Orbits of Circuit Classes

# Orbits

- **Orbit of a polynomial:** For $f \in \mathbb{F}[x_1, \ldots, x_n]$, the orbit of $f$, denoted by $\mathrm{orb}(f)$ is the set $\{f(A\mathbf{x} + \mathbf{b}) : A \in \mathrm{GL}(n, \mathbb{F}) \text{ and } \mathbf{b} \in \mathbb{F}^n\}$.

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$$\begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

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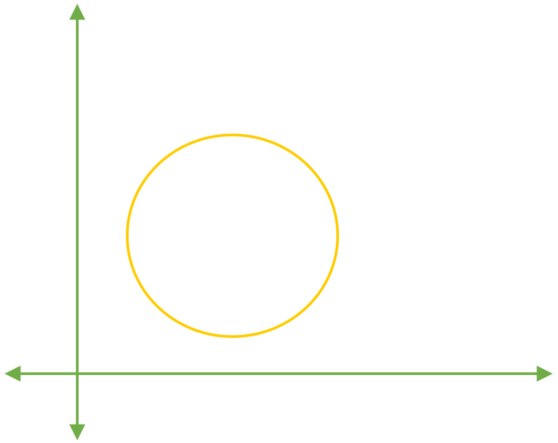
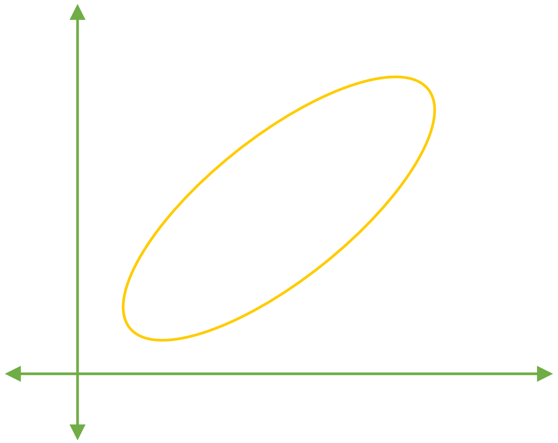


Zero set of $f(\boldsymbol{x})$



Zero set of $g(\boldsymbol{x})$

# Orbits

- **Orbit of a polynomial:** For $f \in \mathbb{F}[x_1, \ldots, x_n]$, the orbit of $f$, denoted by $\mathrm{orb}(f)$ is the set $\{f(A\mathbf{x} + \mathbf{b}) : A \in \mathrm{GL}(n, \mathbb{F}) \text{ and } \mathbf{b} \in \mathbb{F}^n\}$.

- **Orbit of a circuit class:** For a circuit class $\mathcal{C}$, the orbit of $\mathcal{C}$, denoted by $\mathrm{orb}(\mathcal{C})$ is the union of $\mathrm{orb}(f)$ for all $f \in \mathcal{C}$.

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- **Orbit of a circuit class:** For a circuit class $\mathcal{C}$, the orbit of $\mathcal{C}$, denoted by $\mathrm{orb}(\mathcal{C})$ is the union of $\mathrm{orb}(f)$ for all $f \in \mathcal{C}$.

- Recently [MS21, ST21, BG21] studied black-box PIT for orbits of various circuit classes.

MS21: Medini-Shpilka, CCC, 21.
ST21: Saha-Thankey, APPROX-RANDOM, 21.
BG21: Bhargava-Ghosh, APPROX-RANDOM, 21.

# The Power of Orbit Closures

- $\overline{\mathrm{orb}(\mathcal{C})}$ is the set of all polynomials that are "well approximated" by polynomials in $\mathrm{orb}(\mathcal{C})$.

- Ex. 1. $\overline{\mathrm{orb}(\Sigma\Pi)}$ contains <u>depth 3 circuits</u>.

# The Power of Orbit Closures

- $\overline{\mathrm{orb}(\mathcal{C})}$ is the set of all polynomials that are "well approximated" by polynomials in $\mathrm{orb}(\mathcal{C})$.

- Ex. 2. $\overline{\mathrm{orb}(\mathrm{ROF})}$ contains <u>arithmetic formulas</u>.

# The Power of Orbit Closures

- $\overline{\mathrm{orb}(\mathcal{C})}$ is the set of all polynomials that are "well approximated" by polynomials in $\mathrm{orb}(\mathcal{C})$.

- Ex. 3. Iterated Matrix Multiplication $\mathrm{IMM}_{w,d}$.

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the $(1, 1)$-th entry of

$$\begin{bmatrix} x_{1,1,1} & \cdots & x_{1,1,w} \\ \vdots & \ddots & \vdots \\ x_{1,w,1} & \cdots & x_{1,w,w} \end{bmatrix} \cdots\cdots \begin{bmatrix} x_{d,1,1} & \cdots & x_{d,1,w} \\ \vdots & \ddots & \vdots \\ x_{d,w,1} & \cdots & x_{d,w,w} \end{bmatrix}.$$

# The Power of Orbit Closures

- $\overline{\mathrm{orb}(\mathcal{C})}$ is the set of all polynomials that are "well approximated" by polynomials in $\mathrm{orb}(\mathcal{C})$.

- Ex. 3. Iterated Matrix Multiplication $\mathrm{IMM}_{w,d}$.

- Every polynomial computed by a size $s$ <u>formula</u> is in $\overline{\mathrm{orb}\left(\mathrm{IMM}_{3,poly(s)}\right)}$.

- Every polynomial computed by a size $s$ <u>Algebraic Branching Program (ABP)</u> is in $\overline{\mathrm{orb}\left(\mathrm{IMM}_{s,s}\right)}$.

# The Power of Orbit Closures

- PIT for orbit closures of simple models $\Longrightarrow$ PIT for general models like formulas, ABPs, and circuits.

- As a first step, it is natural to try to do PIT for orbits.

# PIT for Orbits

- [KS19] gave polynomial time black box PIT for $\mathrm{orb}(\sum_{i\in[n]} x_i^d)$.
- [MS21] gave polynomial time black box PIT for orbit of the continuant polynomial. Orbit closure of the continuant contains all polynomial sized formulas.

KS19: Koiran-Skomra, CoRR, 19.
MS21: Medini-Shpilka, CCC, 21.

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Trace of

$$\begin{bmatrix} x_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} x_n & 1 \\ 1 & 0 \end{bmatrix}.$$

KS19: Koiran-Skomra, CoRR, 19.
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# PIT for Orbits

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- [MS21] gave quasi-polynomial time black box PIT for $\mathrm{orb}(\Sigma\Pi)$.
- [MS21, ST21] gave quasi-polynomial time black box PIT for $\mathrm{orb}(\mathrm{ROF})$.
- [ST21, BG21] gave quasi-polynomial time black box PIT for orbits of commutative ROABPs and constant width ROABPs computing polynomials with individual degree $O(\log n)$.


KS19: Koiran-Skomra, CoRR, 19.
MS21: Medini-Shpilka, CCC, 21.
ST21: Saha-Thankey, APPROX-RANDOM, 21.
BG21: Bhargava-Ghosh, APPROX-RANDOM, 21.

# Some open problems

- Polynomial time PIT for $\Sigma^{k}\Pi\Sigma\Pi^{\delta}$ circuits by proving the Sylvester-Gallai type conjecture proposed by [Gup14].

- Polynomial time black box PIT for ROABPs.

- Black box PIT for $\mathrm{orb}(\mathrm{IMM}_{w,d})$ and orbits of ROABPs.

Gup14: Gupta, ECCC, 14.

Thank You!