Assignment 5
due on Wednesday, May 31, 2017

Name:

Exercise 1 (10 points).
Let C be a circuit and p be the polynomial computed by C. Prove (for instance by structural induction) that

$$p = \sum_{T \in \mathrm{pt}(C)} w(T).$$

Exercise 2 (5+5+10 points).
Let $(f_n), (g_n) \in \mathrm{VNP}$. Let q be minimal such that $f_n \in \mathbb{F}[X_1, \ldots, X_{q(n)}]$. Let (m_n) be a sequence of monomials such that $\deg m_n$ is polynomially bounded. Prove the following closure properties of VNP.

1. VNP is closed under addition and multiplication, that is, $(f_n + g_n), (f_n g_n) \in \mathrm{VNP}$.
2. VNP is closed under substitutions, that is $(f_n(g_1(X), \ldots, g_{q(n)}(X))) \in \mathrm{VNP}$.
3. VNP is closed under taking coefficients: Consider f_n as a polynomial in the variables that appear in m_n. The coefficients are polynomials in the remaining variables. Let h_n be the coefficient of m_n in f_n. Prove that $(h_n) \in \mathrm{VNP}$.

Exercise 3 (5+5 points).

1. Let
$$s_n = \prod_{i=1}^{n} \sum_{j=1}^{n} X_{i,j} Y_j,$$
considered as a polynomial in the variables Y_j and the coefficients are polynomials in the variables $X_{i,j}$. Prove that the coefficient of $Y_1 Y_2 \ldots Y_n$ is $\mathrm{per}\, X$.
2. Prove that VP is not closed under taking coefficients, unless VP = VNP.

Exercise 4 (10 bonus points).
Use the previous exercise to prove that the permanent has formulas of size $2^n \cdot \mathrm{poly}(n)$. (Note that the standard definition gives a formula of size $n! \cdot (n-1) - 1$.) Convert s_n into an univariate polynomial in only one variable Y instead of $Y_1, \ldots, Y_n$ and recover the appropriate coefficient via fast interpolation or using fast polynomial multiplication.