Assignment 8
due on Wednesday, June 21, 2017

Name:

Exercise 1 (10 points).
Let U, U', V, and V' be finite-dimensional vector spaces. Let $t \in U \otimes V$ and let $A : U \to U'$ and $B : V \to V'$ be linear maps. Prove that $(A \otimes B)(t)$ (as defined in the lecture) is well-defined, that is, if we take two decompositions of t into rank-one tensors, then we will get the same image.

Exercise 2 (5+5 points).
Let U, V, and W be finite-dimensional vector spaces. Prove that tensor product is commutative and associate, that is, there are natural isomorphisms

1. $U \otimes V \cong V \otimes U$ and

2. $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$.

Try to only use the universal property to construct the isomorphisms and not the concrete construction given in the lecture.

Exercise 3 (10 points).
Let $p(X) = \sum_{i=0}^{d} a_i X^i$ and $q(x) = \sum_{i=0}^{d} b_i X^i$ be two univariate polynomials with indeterminates as coefficients. Define the bilinear forms $c_0, \dots, c_{2d}$ in $a_0, \dots, a_d$ and $b_0, \dots, b_d$ by

$$p(X) \cdot q(X) = \sum_{i=0}^{2d} c_i X^i.$$

Write down the corresponding tensor t_d. Prove that $R(t_d) \leq 2d$ if the underlying field has at least $2d+1$ elements. Hint: Use evaluation and interpolation. (Bonus: Can you do it with just $2d$ elements?)

Exercise 4 (5+5+0 points).
Let U, V, and W be finite-dimensional vector spaces. Let $t \in U \otimes V \otimes W$. Let $k = \dim U$.

1. Prove that if $R(t) < k$, then there is a nonzero linear form $a \in U^*$ such that $(a \otimes I_V \otimes I_W)(t) = 0$. Here I_V and I_W are the identity on V and W.

2. Prove that $R(\langle r \rangle) = r$.

3. Prove that $R(t_d) \geq 2d + 1$, where t_d is the tensor of the previous exercise.