Assignment 12
due on Wednesday, July 11, 2018

Name:

Exercise 1 (10 points).
Prove that there exists no decomposition of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix},$$

where the stars are complex constants.

On the other hand, prove that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ can be approximated arbitrarily closely by such decompositions.

Exercise 2 (10 points).
Let U_n denote the group of $n \times n$ upper triangular matrices with 1s on the main diagonal. For $1 \leq i < j \leq n$, $\alpha \in \mathbb{C}$, let $x_{ij}(\alpha) \in U_n$ denote the identity matrix with an entry α in row i and column j. Prove that U_n is generated as a group by the set $\{x_{ij}(\alpha) \mid 1 \leq i < j \leq n, \alpha \in \mathbb{C}\}$.

Exercise 3 (10 points).
Let V be a polynomial GL_n-representation. For $i < j$ let $x_{ij}(\alpha)$ be defined as in Exercise 2. The *raising operator* $E_{ij} : V \to V$ is a map defined via

$$E_{ij}(v) = \lim_{\varepsilon \to 0} \tfrac{1}{\varepsilon}((x_{ij}(\varepsilon)v) - v).$$

Prove that E_{ij} is well-defined, i.e., that the limit exists. Furthermore, prove that $E_{ij} : V \to V$ is a linear map.

Exercise 4 (10 points).
Let V be a polynomial GL_n-representation and let $v \in V$ be of weight λ for $\lambda \in \mathbb{N}^n$. Prove that $E_{ij}(v) = \sum_{\mu \rhd \lambda} v_\mu$, where each v_μ is either zero or a weight vector of weight μ.