Assignment 3
due on Wednesday, May 2, 2018

Name:

Exercise 1 (10 points).
Consider the action of $\mathbb{C}^{N\times N}$ on $\mathbb{C}[X_1,\ldots,X_N]_d$ defined in the lecture. Compute the following polynomial in the standard monomial basis:

$$\begin{pmatrix} 2 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} (X_1X_2^2 + X_3)$$

Exercise 2 (15 points).
Prove that the Waring rank of a polynomial is always finite by proving that

$$X_1X_2\cdots X_d = \frac{1}{d!\,2^{d-1}} \sum_{\alpha\in\{-1,1\}^{d-1}} \left(\prod_{i=1}^{d-1} \alpha_i\right) (\alpha_1X_1 + \alpha_2X_2 + \cdots + \alpha_{d-1}X_{d-1} + X_d)^d.$$

Exercise 3 (15 points).
Let GL_n denote the group of invertible complex $n\times n$ matrices. Let $G = \mathsf{GL}_n \times \mathsf{GL}_n$ and let $V = \mathbb{C}^{n\times n}$. Define an action of G on V by

$$(g_1, g_2)v := g_1 \cdot v \cdot g_2^t,$$

where "$\cdot$" is the product of matrices. Let $v \in V$ have matrix rank $\mathrm{rk}(v) = k$. Prove that

$$Gv = \{w \in V \mid \mathrm{rk}(w) = k\}.$$