Assignment 9
due on Wednesday, June 13, 2018

Name:

On this homework sheet we prove that the Hamiltonian cycle polynomial is VNP-complete under p-projections. We mimic the VNP-completeness proof for the permanent polynomial.

Let $\mathfrak{S}_n$ denote the symmetric group on n symbols. A *Hamiltonian cycle* $\pi \in \mathfrak{S}_n$ is a permutation such that the list $(1, \pi(1), \pi(\pi(1)), \ldots, \pi^{n-1}(1))$ does not have a repeating value. Let $C_n \subseteq \mathfrak{S}_n$ denote the subset of Hamiltonian cycles.

The Hamiltonian cycle polynomial is defined as

$$\mathrm{HC}_n(x_{1,1}, x_{1,2}, \ldots, x_{n,n}) = \sum_{\pi \in C_n} \prod_{i=1}^{n} x_{i,\pi(i)}$$

Exercise 1 (10 points)**.**
Given a layered directed acyclic labeled graph G with source s and sink t. The value $v(G)$ of G is defined as the sum of the values of all s-t-paths, where the value of an s-t-path is defined as the product of the labels of all its edges. Prove that there is graph G' such that $\mathrm{HC}(G') = v(G)$ and that the number of vertices of G' is polynomially bounded in the number of vertices of G. (Here we identified the directed graph G' with its adjacency matrix.)

Hint: Recall that we have seen analogous constructions for the determinant and for the permanent. Here, instead of introducing self-loops, you should connect the vertices within each layer cyclically.

Exercise 2 (5 points)**.**
The following technique is called *vertex splitting.*

Prove that from a directed acyclic labeled graph G you can create a directed acyclic labeled graph G' with one more vertex such that $\mathrm{HC}(G) = \mathrm{HC}(G')$ and in G' there is a vertex that has outdegree 1 and another vertex that has indegree 1.

Exercise 3 (7 points)**.**
A Hamiltonian s-t-path is a path from s to t in a digraph that uses each vertex exactly once.

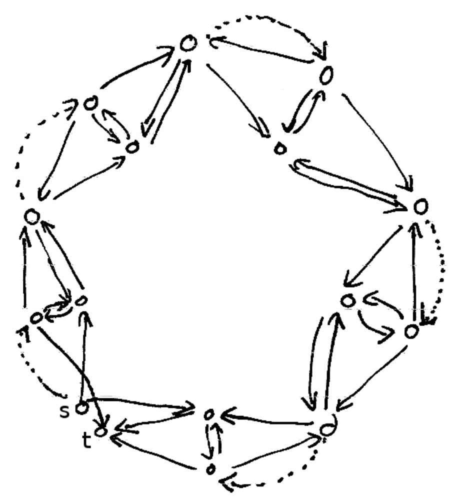
Prove that there exists a directed acyclic "Rosette graph" $R(i)$ with source s and sink t and a set X of so-called connector edges such that

- $|X| = i$,
- there are exactly two Hamiltonian s-t-paths that take no connector edges, and

- for every subset $\emptyset \neq S \subseteq X$ there is a unique Hamiltonian s-t-path that uses exactly the connector edges in X.

The number of vertices shall be polynomially bounded in i.

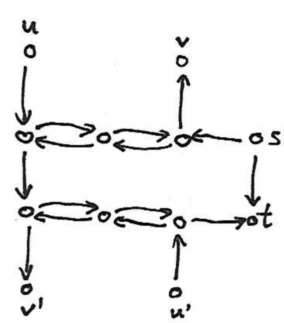
Hint: The following picture for $i = 5$ should help, where the connector edges are dashed edges.



Exercise 4 (8 points)**.**
Given a directed acyclic graph G and a vertex s with outdegree 1 and a vertex t with indegree 1 with an edge (s, t) in G. Moreover, given vertices u, v, u', v' with edges (u, v) and (u', v') in G. Prove that one can replace these three edges with a constant size subgraph H (the "equality gadget") and obtain a new graph G' such that

- there is a bijection between {Hamiltonian paths in G'} and {Hamiltonian paths in G that either use both (u, v) and (u', v') or use neither (u, v) nor (u', v')}.

Hint: The following picture should help.



Exercise 5 (10 points)**.**
Use the previous exercises to show the VNP-hardness of HC_n under p-projections.

Hint: Add Rosette graphs and connect the connector edges with their counterparts in G using the equality gadget as in the proof of the VNP-hardness of the permanent. For each new subgraph that you add, split some vertex once using Exercise 2.