Assignment 1
due on Tuesday, October 24, 2017

Name:

Exercise 1 (10 points).
Let K_n denote the *continuant*, which is the $(1,1)$-entry of the product

$$\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix}.$$

What is the coefficient of the monomial $x_{i_1} x_{i_2} \cdots x_{i_\ell}$ in K_n?

Exercise 2 (10 points).
Prove that $K_n(x_1, x_2, \ldots, x_n) = K_n(x_n, x_{n-1}, \ldots, x_1)$.

Exercise 3 (10 points).
Prove that

$$K_n = \det \begin{pmatrix} x_1 & 1 & 0 & \cdots & 0 \\ -1 & x_2 & 1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \ldots & 0 & -1 & x_n \end{pmatrix}$$

Exercise 4 (10 points).
For a homogeneous degree m polynomial h we define $L(h)$ to be the smallest n such that $x_1^{n-m} h \in \overline{\mathsf{GL}_n K_n}$ (as usual, the variables in h are ordered consecutively). In the lecture we showed that a sequence (h_m) is in $\overline{\mathrm{VP}_\mathrm{e}}$ iff its sequence $L(h_m)$ is polynomially bounded. Prove that this is still true if we replace K_n by any of the two polynomials

$$K_n' = \det \begin{pmatrix} x_1 & 1 & 0 & \cdots & 0 \\ 1 & x_2 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \ldots & 0 & 1 & x_n \end{pmatrix}$$

or

$$K_n'' = \mathrm{tr}\left(\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix} \right).$$