Assignment 10
due on Tuesday, Jan 16, 2018

Name:

For $i, j \in \mathbb{N}$ and a partition $\lambda \vdash_i ij$ let $\{\lambda\}^{i\times j}$ denote the $(i \times j)$-weight space in the irreducible GL_i-representation $\{\lambda\}$. For $\nu \vdash i$ we define the generalized plethysm coefficient $a_\lambda(\nu, j)$ via embedding $\mathfrak{S}_i \hookrightarrow \mathsf{GL}_i$ and decomposing:

$$\{\lambda\}^{i\times j} = \bigoplus_{\nu \vdash i} a_\lambda(\nu, j)[\nu].$$

Exercise 1 (5 points)**.**
Prove that

$$a_\lambda(\nu, 1) = \begin{cases} 1 & \text{if } \lambda = \nu, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 2 (10 points)**.**
For $i, j \in \mathbb{N}$ the space $\bigotimes^i \mathrm{Sym}^j V$ has a canonical action of $\mathfrak{S}_i \times \mathsf{GL}(V)$. Prove that as an $(\mathfrak{S}_i \times \mathsf{GL}(V))$-representation we have

$$\bigotimes^i \mathrm{Sym}^j V = \bigoplus_{\substack{\lambda \vdash ij \\ \nu \vdash i}} a_\lambda(\nu, j)\ [\nu] \otimes \{\lambda\}.$$

Exercise 3 (10 points)**.**
For partitions $\nu^1, \dots, \nu^m$ and $\lambda \vdash \sum_{i=1}^m |\nu^i|$ let $c^\lambda_{\nu^1,\dots,\nu^m}$ denote the multiplicity of the irreducible $\mathsf{GL}(V)$-representation $\{\lambda\}$ in the tensor product $\{\nu^1\} \otimes \cdots \otimes \{\nu^m\}$ of irreducible $\mathsf{GL}(V)$-representations. This is called the multi-Littlewood-Richardson coefficient. If $m = 2$, then this is the classical Littlewood-Richardson coefficient. Prove that

$$c^\lambda_{\nu^1,\dots,\nu^m} = \sum_{\substack{\mu^1,\dots,\mu^{m-2} \\ \mu^i \vdash \sum_{j=1}^{i+1} |\nu^j|}} c^{\mu^1}_{\nu^1,\nu^2} \cdot c^{\mu^2}_{\mu^1,\nu^3} \cdot c^{\mu^3}_{\mu^2,\nu^4} \cdots c^\lambda_{\mu^{m-2},\nu^m}.$$

Exercise 4 (15 points)**.**
Embed $\mathfrak{S}_i \times \mathfrak{S}_j \hookrightarrow \mathfrak{S}_{i+j}$ as a Young subgroup (i.e., $\mathfrak{S}_i$ acts on $\{1, \dots, i\}$ and $\mathfrak{S}_j$ acts on $\{i+1, \dots, i+j\}$). Use Schur-Weyl duality to prove that given a partition $\lambda \vdash i + j$ the irreducible $\mathfrak{S}_{i+j}$-representation $[\lambda]$ decomposes as an $\mathfrak{S}_i \times \mathfrak{S}_j$-representation as follows:

$$[\lambda] = \bigoplus_{\substack{\mu \vdash i \\ \nu \vdash j}} c^\lambda_{\mu,\nu}([\mu] \otimes [\nu]).$$