Assignment 11
due on Tuesday, Jan 23, 2018

Name:

Exercise 1 (5 + 5 + 5 points)**.**
(Grenet's construction) Consider the following edge-weighted graph G_n: The nodes are all subsets of $\{1, \ldots, n\}$, but we identify $\emptyset$ with $\{1, \ldots, n\}$, so G_n has $2^n - 1$ nodes. There is an edge from S to T with weight $X_{i,j}$ if $|S| = i-1$ and $T = S \cup \{j\}$. The node $\emptyset$ will have outgoing edges with weight $X_{1,j}$ to the node $\{j\}$, $1 \leq j \leq n$ and incoming edges with weights $X_{n,j}$ from the node $\{1, \ldots, n\} \setminus \{j\}$,. Furthermore, every node except $\emptyset$ gets a self loop of weight 1.

1. Prove that the cycle covers of G_n stand in one-to-one correspondence with the permutations of $\mathfrak{S}_n$.
2. Prove that $\mathrm{per}(G_n) = \pm \det(G_n)$.
3. Prove that $\mathrm{dc}(\mathrm{per}_n) \leq 2^n - 1$.

Exercise 2 (5+5 points)**.**

1. Prove that $\mathrm{dc}(\mathrm{per}_2) = 2$.
2. Prove that the $\mathrm{per}_2 \not\leq_p \det_2$ but $\mathrm{per}_2 \leq_p \det_3$.

Alper, Bogart and Velasco prove that $\mathrm{dc}(\mathrm{per}_3) = 7$. The exact value of $\mathrm{dc}(\mathrm{per}_4)$ is currently unknown to my best knowledge.

Exercise 3 (15 points)**.**
The Hamilton cycle polynomial HC_n is defined like the permanent, but we do not sum over all permutations but only over permutations that are n-cycles. Prove that $\mathrm{dc}(\mathrm{HC}_n) \leq (n-1)2^{n-2} + 1$. (It might be helpful to prove that there is a bijection between permutations in $\mathfrak{S}_n$ and $(n+1)$-cycles in $\mathfrak{S}_{n+1}$.)