Assignment 7
due on Tuesday, Dec 19, 2017

Name:

Let T_{SL_n} denote the group of diagonal matrices with determinant 1. Consider the group $G := T_{\mathsf{SL}_n} \times T_{\mathsf{SL}_n}$, which acts on $\mathrm{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n)$ and preserves both the determinant and the permanent. The group $\mathfrak{S}_n \times \mathfrak{S}_n$ acts on the space of G-invariants $V := (\mathrm{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n))^G$ and we saw in the lecture that per_n is the unique polynomial (up to scale) of type $((n),(n))$ in V, whereas $\det_n$ is the unique polynomial (up to scale) of type $((1^n),(1^n))$ in V.

In the following exercises, consider $W := (\mathrm{Sym}^{2n}(\mathbb{C}^n \otimes \mathbb{C}^n))^G$. Analogously to V, there is a canonical action of $\mathfrak{S}_n \times \mathfrak{S}_n$ on W.

Exercise 1 (4 points).
Prove that in W there are nonzero polynomials of type $((n),(n))$.

Exercise 2 (4 points).
Let $n > 1$ and prove that in W the polynomials of type $((n),(n))$ are not unique up to scale.

Exercise 3 (4 points).
Prove that in W there are nonzero polynomials of type $((1^n),(1^n))$.

Exercise 4 (12 points).
Let $n = 3$ and prove that in W the polynomials of type $((1^3),(1^3))$ are not unique up to scale.

Exercise 5 (16 points).
Prove that for $n = 6$ the space W contains a nonzero polynomial of type $((6),(1^6))$.

You can use that the GL_6-representation $\mathrm{Sym}^6(\mathrm{Sym}^2\,\mathbb{C}^6)$ decomposes as

$$\{(12)\}+\{(10,2)\}+\{(8,4)\}+\{(8,2^2)\}+\{(6^2)\}+\{(6,4,2)\}+\{(6,2^3)\}+\{(4^3)\}+\{(4^2,2^2)\}+\{(4,2^4)\}+\{(2^6)\}$$

and that the GL_6-representation $\bigwedge^6(\mathrm{Sym}^2\,\mathbb{C}^6)$ decomposes as

$$\{(7,1^5)\} + \{(6,3,1^3)\} + \{(5,4,2,1)\} + \{(4^3)\}.$$