Assignment 8, Christmas break sheet
due on Tuesday, Jan 2, 2018

Name:

Exercise 1 (10 points)**.**
Prove that every complex matrix of finite order is diagonalizable.

Exercise 2 (10 points)**.**
Fix a natural number n. Given a list $(c_1, c_2, \dots, c_\ell) \in \{1, \dots, n\}^\ell$ of pairwise distinct numbers, the corresponding *cycle* is the permutation π that satisfies $\pi(c_i) = c_{i+1}$ for all $1 \le i \le \ell - 1$, $\pi(c_\ell) = c_1$, and $\pi(j) = j$ if $\forall i\colon j \neq c_i$. The number ℓ is called the *length* of the cycle and the set $\{c_1, \dots, c_\ell\}$ is called its support. Two cycles are called *disjoint* if their supports have empty intersection. Clearly disjoint cycles commute. Prove that every $\pi \in \mathfrak{S}_n$ can be written uniquely (up to a permutation of the factors) as a product of disjoint cycles.

This is called the *cycle decomposition*.

Exercise 3 (10 points)**.**
Let $\pi = c^{(1)}c^{(2)} \cdots c^{(k)} \in \mathfrak{S}_n$ be the cycle decomposition. Let ℓ_i denote the length of the cycle $c^{(i)}$. If we sort the list of lengths $(\ell_1, \ell_2, \dots, \ell_k)$, then we obtain a partition $\lambda \vdash n$ that we call the *cycle type* of π. Prove that two permutations are in the same conjugacy class iff they have the same cycle type.

Exercise 4 (10 points)**.**
Let G be a finite group and let V and W be two G-representations (in particular, V and W are finite dimensional). Then the direct sum $V \oplus W$ of vector spaces is a G-representation via

$$g(v, w) := (gv, gw).$$

Prove that the character $\chi_{V\oplus W}$ satisfies $\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g)$.