max planck institut informatik

SAARLAND UNIVERSITY

COMPUTER SCIENCE

Prof. Dr. Markus Bläser and Dr. Christian Ikenmeyer **Winter 2017/2018**

Exercises for Geometric complexity theory 2

https://people.mpi-inf.mpg.de/~cikenmey/teaching/winter1718/gct2/index.html

Exercise sheet 1 Solutions Due: **Tuesday, October 24, 2017**

Total points : 40

Exercise 1 (10 Points). Let K_n denote the continuant, which is the $(1,1)$-entry of the product

$$\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix}$$

What is the coefficient of the monomial $x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_\ell}$ in K_n ?

Solution 1. For this, we can look at the ABP computing K_n. By looking at this ABP, we note that if a monomial $x_{i_1}x_{i_2}\ldots x_{i_\ell}$ appears in K_n and a variable x_k is not present in $x_{i_1}x_{i_2}\ldots x_{i_\ell}$ then at least one of x_k or x_{k-1} must also be missing from $x_{i_1}x_{i_2}\ldots x_{i_\ell}$. Moreover, this is a sufficient condition also for a monomial $x_{i_1}x_{i_2}\ldots x_{i_\ell}$ to appear in K_n. Thus $x_{i_1}x_{i_2}\ldots x_{i_\ell}$ appears in K_n iff $x_{i_1}x_{i_2}\ldots x_{i_\ell}$ can be obtained by $x_1 \cdot x_2 \cdot \cdots \cdot x_n$ by removing disjoint pairs of consecutive variables. Also, it is easy to observe that all the monomials in K_n have coefficient one. Thus the coefficient of the monomial $x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_\ell}$ in K_n is 1 iff $x_{i_1}x_{i_2}\ldots x_{i_\ell}$ can be obtained from $x_1 \cdot x_2 \cdot \cdots \cdot x_n$ by removing disjoint pairs of consecutive variables. Otherwise this coefficient is zero.

Exercise 2 (10 Points). Prove that $K_n(x_1, x_2, \ldots, x_n) = K_n(x_n, x_{n-1}, \ldots, x_1)$.

Solution 2 (Typeset by all students in the lecture)**.** This directly follows since the $(1,1)$ entry does not change under transposing a matrix and $(AB)^t = B^tA^t$. Thus we have

$$\begin{aligned}
K_n(x_1, x_2, \ldots, x_n) &= \left(\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix}\cdot\ldots\cdot\begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix}\right)_{(1,1)} \\
&= \left(\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix}\cdot\ldots\cdot\begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix}\right)^t_{(1,1)} \\
&= \left(\begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix}^t\cdot\begin{pmatrix} x_{n-1} & 1 \\ 1 & 0 \end{pmatrix}^t\cdot\ldots\cdot\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix}^t\right)_{(1,1)} \\
&= \left(\begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} x_{n-1} & 1 \\ 1 & 0 \end{pmatrix}\cdot\ldots\cdot\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix}\right)_{(1,1)} = K_n(x_n, x_{n-1}, \ldots, x_1)
\end{aligned}$$

One can also prove $K_n(x_1, x_2, \dots, x_n) = K_n(x_n, x_{n-1}, \dots, x_1)$ by reversing every edge in the ABP computing K_n.

Remark by Ikenmeyer: One can also use Exercise 1: Adjacency of variables does not change when we reverse the order.

Exercise 3 (10 Points)**.** Prove that

$$K_n = \det \begin{pmatrix} x_1 & 1 & 0 & \dots & 0 \\ -1 & x_2 & 1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & -1 & x_n \end{pmatrix}.$$

Solution 3. In order to prove this we will first prove the following claim:

$$K_n(x_1, x_2, \dots, x_n) = x_n \cdot K_{n-1}(x_1, x_2, \dots, x_{n-1}) + K_{n-2}(x_1, x_2, \dots, x_{n-2}).$$

This is easy to see by looking at the ABP computing K_n or by a short calculation using the definition of K_n as the (1,1)-entry of the product of matrices.

Now the actual exercise can be solved by induction. The base case for K_2 is trivial since

$$\det \begin{pmatrix} x_1 & 1 \\ -1 & x_2 \end{pmatrix} = x_1 x_2 + 1$$

For the induction step we use Laplace to compute the determinant:

$$\begin{aligned} \det \begin{pmatrix} x_1 & 1 & 0 & \dots & 0 \\ -1 & x_2 & 1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & -1 & x_{n+1} \end{pmatrix} &= \\ x_{n+1} \cdot \det \begin{pmatrix} x_1 & 1 & 0 & \dots & 0 \\ -1 & x_2 & 1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & -1 & x_n \end{pmatrix} &- (-1) \cdot \det \begin{pmatrix} x_1 & 1 & 0 & \dots & 0 \\ -1 & x_2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & -1 & x_{n-1} & 1 \end{pmatrix} \\ &= x_{n+1} \cdot \det A_n + \det A_{n-1} \end{aligned}$$

where A_n denotes the matrix that is given in the exercise. Thus $\det A_n$ satisfies the same recursion formula as K_n which concludes the proof.

Exercise 4 (10 Points)**.** For a homogeneous degree m polynomial h we define $L(h)$ to be the smallest n such that $x_1^{n-m} h \in \overline{\mathsf{GL}_n K_n}$ (as usual, the variables in h are ordered consecutively).

In the lecture we showed that a sequence (h_m) is in $\overline{\mathsf{VP}_e}$ iff its sequence $L(h_m)$ is polynomially bounded. Prove that this is still true if we replace K_n by any of the two polynomials

$$K'_n = \det \begin{pmatrix} x_1 & 1 & 0 & \dots & 0 \\ 1 & x_2 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 & x_n \end{pmatrix}$$

or

$$K''_n = \operatorname{tr}\left(\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix}\right)$$

Solution 4. For this, we need to show that K'_n and K''_n can be used to approximate any $(h_m) \in \overline{\mathsf{VP}_e}$ with $n \le \text{poly}(m)$, and vice-versa. As usual, we use the notation $i = \sqrt{-1}$. Now we show that $i^n \cdot K_n$ reduces to K'_n under suitable reductions.

Note that $K'_n(i \cdot x_1, i \cdot x_2, \dots, i \cdot x_n) = i^n \cdot (K_n(x_1, x_2, \dots, x_n))$. This can be proved by induction on n and using the recurrence for K_n, which was derived in the solution of exercise 3. Thus if (h_m) can be approximated by $K_{\text{poly}(m)}$ then it can also be approximated by $K'_{\text{poly}(m)}$. Moreover, this reduction from K_n to K'_n which we showed above can be easily modified to work in other direction also. Thus (h_m) can be approximated by $K_{\text{poly}(m)}$ iff (h_m) can be approximated by $K'_{\text{poly}(m)}$. Thus $(h_m) \in \overline{\mathsf{VP}_e}$ iff (h_m) can be approximated by $K'_{\text{poly}(m)}$.

Now we prove that $K''_{\text{poly}(m)}$ can be used to approximate any $(h_m) \in \overline{\mathsf{VP}_e}$. This directly follows from Proposition 3.6 from the paper "On algebraic branching programs of small width", see Proposition 3.6 in `https://arxiv.org/pdf/1702.05328.pdf`. Moreover we also know that $(K''_n) \in \mathsf{VP}_e$. Thus $(h_m) \in \overline{\mathsf{VP}_e}$ iff (h_m) can be approximated by $K''_{\text{poly}(m)}$.