max planck institut informatik

SAARLAND UNIVERSITY
COMPUTER SCIENCE

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Exercises for Geometric complexity theory 2

https://people.mpi-inf.mpg.de/~cikenmey/teaching/winter1718/gct2/index.html

Exercise sheet 2 Solutions Due: **Tuesday, November 7, 2017**

Total points : 60

For three partitions λ, μ, ν of d let $k(\lambda, \mu, \nu)$ denote the Kronecker coefficient, i.e., the dimension of the $\mathfrak{S}_d$-invariant space $\dim([\lambda] \otimes [\mu] \otimes [\nu])^{\mathfrak{S}_d}$.

Exercise 1 (10 Points)**.** Determine $k((2,1),(2,1),(2,1))$.

Solution 1. We know that $\dim([2,1] \otimes [2,1] \otimes [2,1]) = 8$. By computation, we can show that if symmetrize all the basis vectors of $[2,1] \otimes [2,1] \otimes [2,1]$ over $\mathfrak{S}_3$, then the image of all such symmetrizations is spanned by the following vector.

$$
\begin{aligned}
-2 \cdot \ytableaushort{12,3} \otimes \ytableaushort{12,3} \otimes \ytableaushort{12,3} &+ \\
\ytableaushort{12,3} \otimes \ytableaushort{12,3} \otimes \ytableaushort{13,2} &+ \\
\ytableaushort{12,3} \otimes \ytableaushort{13,2} \otimes \ytableaushort{12,3} &+ \\
\ytableaushort{12,3} \otimes \ytableaushort{13,2} \otimes \ytableaushort{13,2} &+ \\
\ytableaushort{13,2} \otimes \ytableaushort{12,3} \otimes \ytableaushort{12,3} &+ \\
\ytableaushort{13,2} \otimes \ytableaushort{12,3} \otimes \ytableaushort{13,2} &+ \\
\ytableaushort{13,2} \otimes \ytableaushort{13,2} \otimes \ytableaushort{12,3} &+ \\
-2 \cdot \ytableaushort{13,2} \otimes \ytableaushort{13,2} \otimes \ytableaushort{13,2} &+
\end{aligned}
$$

which is clearly nonzero. This implies that $k((2,1),(2,1),(2,1)) = 1$.

Exercise 2 (10 Points)**.** Let $\lambda \vdash d$. Let λ^t denote the transpose Young diagram of λ . Let $d \times 1$ denote the column partition. Prove that $k(d \times 1, \lambda, \lambda^t) \leq 1$.

20 bonus points if you also prove $k(d \times 1, \lambda, \lambda^t) = 1$.

Solution 2 (Typeset by all students in the lecture)**.** We can again look at the dot diagrams. The $(d \times 1)$ shape is only a single hyperedge and thus is uninteresting. We can now group points directly by λ without loss of generality. λ^t now only has one way to seperate the points inside those groups, as every new seperation has to span all groups given by λ. Since this is unique we know that $k(d \times 1, \lambda, \lambda^t) \leq 1$.

Remark by Ikenmeyer: This is correct, but formally the pigeonhole principle is applied several times to see that every new separation has to span all groups.

Exercise 3 (20 Points)**.** Let λ, μ, ν be partitions of d and let $\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}$ be partitions of $\tilde{d}$. Prove the "semi-group property": If $k(\lambda, \mu, \nu) > 0$ and $k(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}) > 0$, then $k(\lambda + \tilde{\lambda}, \mu + \tilde{\mu}, \nu + \tilde{\nu}) \geq \max\{k(\lambda, \mu, \nu), k(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu})\}$, where the addition of partitions is defined as adding row-lengths of the corresponding Young diagrams.

Hint: Look at the analogous proof for plethysm coefficients

Solution 3. Note that $\otimes^3\mathbb{C}^n$ is an irreducible variety for all n. Now apply Proposition 19.6.6 (The semi-group property) from https://people.mpi-inf.mpg.de/∼cikenmey/teaching/summer17/introtogct/gct.pdf

to $Z = \mathbb{A} = \otimes^3\mathbb{C}^n$ for some $n \geq d + \tilde{d}$.

Exercise 4 (20 Points)**.** Let (i) denote the partition that only has a single row. Let λ, μ, ν be partitions of d. As a preliminary task, prove that the sequence

$$K_i := k(\lambda + (i), \mu + (i), \nu + (i))$$

is monotonously non-decreasing. Then prove that the sequence K_i stabilizes, i.e., $\exists i_0 \forall i \geq i_0 : K_i = K_{i_0}$.

Solution 4 (Partially typeset by all students in the lecture)**.** To prove the first case we do a case distinction. Namely the cases where $k(\lambda, \mu, \nu) > 0$ and $k(\lambda, \mu, \nu) = 0$ respectively. For the first case, we can just apply exercise 3.

In the second case, either K_i always is zero then it is trivially monotonously non-decreasing. Otherwise $K_j > 0$ for some j. Then we can again apply exercise 3 to get that the sequence K_i is monotonously non-decreasing.

Thus we have proven that the sequence is non-decreasing.

Now to prove the stabilising of the Kronecker coefficient we can look at our dot diagrams again. We can see that $\lambda + (i)$ has at most $|\lambda|$ dots connected in non singleton groups. So dot diagrams for $(\lambda + (i), \mu + (i), \nu + (i))$ together can only connect at most $|\lambda| + |\mu| + |\nu|$ dots into non-singleton groups. Thus the the choices are bound irrespective of i and we get there is a total upper bound on $k(\lambda + (i), \mu + (i), \nu + (i))$ resulting in K_i stabilizing.