Prof. Dr. Markus Bläser and Dr. Christian Ikenmeyer **Winter 2017/2018**

Exercises for Geometric complexity theory 2

https://people.mpi-inf.mpg.de/~cikenmey/teaching/winter1718/gct2/index.html

Exercise sheet 3 Solutions   Due: **Tuesday, November 14, 2017**

Total points : 40

As in the lecture, to each obstruction design $\mathcal{H}_n$ we have the corresponding highest weight vector function $f_{\mathcal{H}_n}$.

Exercise 1 (10 Points)**.** Consider two obstruction designs $\mathcal{H}_1$ and $\mathcal{H}_2$. Take their union (the usual union of hyper-graphs by just drawing them next to each other) and call the resulting obstruction design $\mathcal{H}_3$. Prove that

$$f_{\mathcal{H}_3} = f_{\mathcal{H}_1} \cdot f_{\mathcal{H}_2}$$

as functions.

Solution 1 (Typeset by all students in the lecture)**.** We have the following definition for $f_{\mathcal{H}}$:

$$\begin{aligned} f_{\mathcal{H}}(w) &= \sum_{J:\mathcal{H}\to\tau} \operatorname{eval}_{\mathcal{H}}(J) = \sum_{J:\mathcal{H}\to\tau} \prod_{k=1}^{3} \operatorname{eval}_{E^{(k)}}(J^{(k)}) \\ &= \sum_{J:\mathcal{H}\to\tau} \prod_{k=1}^{3} \prod_{e\in E^{(k)}} \det J^{(k)}|e \end{aligned}$$

Now we get:

$$f_{\mathcal{H}_1} \cdot f_{\mathcal{H}_2} = \Big(\sum_{J_1:\mathcal{H}_1\to\tau} \prod_{k=1}^{3} \prod_{e_1\in E_1^{(k)}} \det J_1^{(k)}|e_1 \Big) \cdot \Big(\sum_{J_2:\mathcal{H}_2\to\tau} \prod_{l=1}^{3} \prod_{e_2\in E_2^{(l)}} \det J_2^{(l)}|e_2 \Big)$$

Now since $\mathcal{H}_1$ and $\mathcal{H}_2$ are disjoint sets we can combine both functions to a new function $J : \mathcal{H}_3 \to \tau$ where $\mathcal{H}_3 = \mathcal{H}_1 \cup \mathcal{H}_2$. Here J_3 basically covers all possible pairs of J_1 and J_2.

Now we can rewrite the product above as follows:

$$\sum_{J_3:\mathcal{H}_3\to\tau} \Big(\prod_{k=1}^{3} \prod_{e_1\in E_1^{(k)}} \det J_1^{(k)}|e_1 \Big) \cdot \Big(\prod_{l=1}^{3} \prod_{e_2\in E_2^{(l)}} \det J_2^{(l)}|e_2 \Big)$$

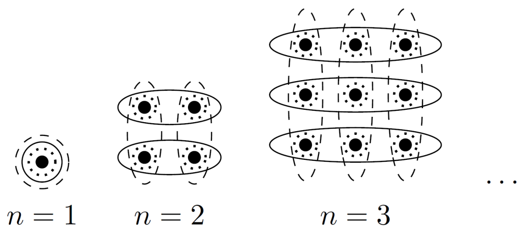
Now each summand has basically 6 factors each (where each of these factors is again a product of determinants, but we do not care about that right now). In the next step we just divide these 6 factors into 3 groups which consists of pairs. Doing that we obtain:

$$\sum_{J_3:\mathcal{H}_3\to\tau}\prod_{k=1}^{3}[(\prod_{e_1\in E_1^{(k)}}\det J_1^{(k)}|e_1)\cdot(\prod_{e_2\in E_2^{(k)}}\det J_2^{(l)}|e_2)]$$

Again we just the fact that $\mathcal{H}_1$ and $\mathcal{H}_2$ were disjoint obstruction designs to combine $E_1^{(k)}$ and $E_2^{(k)}$ into $E_3^{(k)}$ which then gives us:

$$\sum_{J_3:\mathcal{H}_3\to\tau}\prod_{k=1}^{3}\prod_{e\in E_3^{(k)}}\det J_3^{(k)}|e = f_{\mathcal{H}_3}$$

Exercise 2 (15 Points)**.** Consider the following sequence of obstruction designs $\mathcal{H}_n$.



Prove that $f_{\mathcal{H}_n}$ is the zero function if $n > 1$ is odd.

Solution 2 (Typeset by all students in the lecture)**.** We can directly translate the obstruction design into the following tensor of young tableaux (in this case the example of $n = 3$):

1	2	3	4	5	6	7	8	9

$\otimes$

1	2	3
4	5	6
7	8	9

$\otimes$

1	4	7
2	5	8
3	6	9

Note that the values in the columns of the second tableaux are exactly the rows in the third one and vice-versa.

If we now symmetrize over the subgroup of $\mathfrak{S}_{n^2}$ given by swapping two fixed rows in the second tableaux we get 0, because the first tableaux is always invariant, the second one will get a negative sign since we swap an odd number of elements inside the columns and the third one stays invariant again by just swapping back the columns. Thus symmetrizing over $\mathfrak{S}_{n^2}$ will also yield the $\mathfrak{S}_{n^2}$ tensor so $f_{\mathcal{H}_n}$ is also the zero function if $n > 1$ is odd.

Exercise 3 (15 Points)**.** For odd n, find an obstruction design $\mathcal{H}'_n$ of the same type as $\mathcal{H}_n$ in the previous exercise, but with $f_{\mathcal{H}'_n} \neq 0$.

Solution 3. We can achieve this by making the dashed and the continuous hyper-edges the same. If we now evaluate we get:

$$f_{\mathcal{H}'_n}(w) = \sum_{J:\mathcal{H}\to\tau}\prod_{e\in E^{(1)}}\det J^{(1)}|e\cdot\prod_{e\in E^{(2)}}\det J^{(2)}|e\cdot\prod_{e\in E^{(3)}}\det J^{(3)}|e$$

If we evaluate $f_{\mathcal{H}'_n}(w)$ on $w = \sum_{i=1}^n e_i \otimes e_i \otimes g e_i$ for some $g \in \mathsf{GL}_n(\mathbb{R})$, then $\prod_{e \in E^{(2)}} \det J^{(2)}|e$ and $\prod_{e \in E^{(3)}} \det J^{(3)}|e$ always evaluate to the same real number and thus $\prod_{e \in E^{(2)}} \det J^{(2)}|e \cdot \prod_{e \in E^{(3)}} \det J^{(3)}|e$ is always non-negative. If we assume that the first row of g is positive, $\prod_{e \in E^{(1)}} \det J^{(1)}|e$ is always positive.

It is also easy to observe that there are terms $\prod_{e \in E^{(1)}} \det J^{(1)}|e \cdot \prod_{e \in E^{(2)}} \det J^{(2)}|e \cdot \prod_{e \in E^{(3)}} \det J^{(3)}|e$ in above sum which evaluate to non-zero. Thus $f_{\mathcal{H}'_n}(w) \neq 0$.