Prof. Dr. Markus Bläser and Dr. Christian Ikenmeyer **Winter 2017/2018**

Exercises for Geometric complexity theory 2

https://people.mpi-inf.mpg.de/~cikenmey/teaching/winter1718/gct2/index.html

Exercise sheet 4 Solutions Due: **Tuesday, November 21, 2017**

Total points : 40

Exercise 1 (10 Points)**.** Let $X \in K^{m\times m}$, $Y \in K^{m\times n}$, $Z \in K^{n\times m}$, and $W \in K^{n\times n}$ with $\det(W) \neq 0$. Prove that

$$\det\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \det(W)\det(X - YW^{-1}Z)$$

Solution 1 (Typeset by all students in the lecture)**.**

$$\begin{aligned}
\det\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} &= \det\left(\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \cdot \begin{pmatrix} I_m & 0 \\ -W^{-1}Z & I_n \end{pmatrix}\right) \\
&= \det\begin{pmatrix} X - YW^{-1}Z & Y \\ 0 & W \end{pmatrix} = \det(W)\det(X - YW^{-1}Z)
\end{aligned}$$

Exercise 2 (20 Points)**.** In the proof of Strassen's lower bound for 3-slice tensors, we used the tensor $T_A^\wedge$. What happens if we use T_A^S (defined by projecting onto S^2A) instead?

Solution 2 (Typeset by all students in the lecture)**.** First, we determine T_A^S similarly to $T_A^\wedge$ and obtain the following matrix (up to some permutations):

$$\begin{pmatrix} X_1 & 0 & 0 \\ X_2 & X_1 & 0 \\ X_3 & 0 & X_1 \\ 0 & X_2 & 0 \\ 0 & 0 & X_3 \end{pmatrix}$$

It is immediately obvious that $\mathrm{rk}(T_A^S) \geq \mathrm{rk}(X_1) + \mathrm{rk}(X_2) + \mathrm{rk}(X_3)$, as the three column blocks are linearly independent of each other and the rank of each column block is lower bounded by the rank of the X_i.

If T is now given by the slices Id, X and Y with X and Y of full rank, we get that $\mathrm{rk}(T_A^S) = 3\bar{b}$, as the rank cannot be more than full column rank.

If we now had an equivalent of Lemma 2 from the lecture, that is, $R((a \otimes b \otimes c)_A^S) \leq \bar{a} - 1$, we could prove Strassen's theorem and get a nontrivial lower bound for matrix multiplication. But exactly this lemma doesn't hold any more and if we look at the tensor $(a \otimes b \otimes c)_A^S$ as a map

$$\begin{aligned} (a \otimes b \otimes c)_A^S : & A \otimes B^* \to S^2 A \otimes C \\ & a_i \otimes \beta \mapsto \beta(b) \cdot a_i \vee a \otimes c \end{aligned}$$

the dimension of its image doesn't have to be $< \bar{a}$ any more (take for example $(a_1 \otimes b_1 \otimes c_1)$, then the $a_i \vee a$ are linearly independent and thus the dimension of the image is at least $\bar{a}$).

Thus we cannot show a lower bound on matrix multiplication if we try to apply the same techniques as with the slice produced by the alternating group and need a completely different approach.

Exercise 3 (10 Points)**.** Generalize Strassen's lower bound: If $T = (I, X, Y)$ is a 3-slice tensor with slices in $K^{b \times b}$, then

$$\underline{R}(T) \geq b + \frac{1}{2} \cdot \mathrm{rk}(XY - YX).$$

Solution 3 (Typeset by all students in the lecture)**.** In the following we will use the facts that we know about the structure of $T_A^\wedge$ from the lecture where we replace X_1 by the identity matrix Id, X_2 by X and X_3 by Y. This gives us:

$$\mathrm{rk}(T_A^\wedge) = \mathrm{rk} \begin{pmatrix} 0 & Y & -X \\ X & -\mathrm{Id} & 0 \\ -Y & 0 & \mathrm{Id} \end{pmatrix}$$

Now for the following steps we proceed in the same way as seen in Exercise 1 with

$$W = \begin{pmatrix} -\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

Thus we get

$$\mathrm{rk}(T_A^\wedge) = \mathrm{rk} \begin{pmatrix} XY - YX & (Y, -X) \\ 0 & \begin{pmatrix} -\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \end{pmatrix}$$

Now the rank of this matrix is just the sum of the ranks of the two rows of the matrix. The rank of the second row corresponds to the rank of $\begin{pmatrix} -\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$ which is $2 \cdot b$ while the rank of the first row is $\geq$ to the rank of $XY - YX$.

Thus we get

$$\mathrm{rk}(T_A^\wedge) \geq 2b + \mathrm{rk}(XY - YX)$$

which concludes the proof.