Prof. Dr. Markus Bläser and Dr. Christian Ikenmeyer **Winter 2017/2018**

Exercises for Geometric complexity theory 2

https://people.mpi-inf.mpg.de/~cikenmey/teaching/winter1718/gct2/index.html

Exercise sheet 5 Solutions Due: **Tuesday, November 28, 2017**

Total points : 40

Exercise 1 (15 Points). Let $A = M \otimes N^*$, $B = N \otimes L^*$, and $C = L \otimes M^*$ with $\dim M = m$, $\dim N = n$, and $\dim L = \ell$, that is, A can be viewed as the vector space of $m \times n$-matrices, etc. Let $\langle m, n, \ell \rangle \in A \otimes B \otimes C$ be the tensor of the multiplication of $m \times n$-matrices with $n \times \ell$-matrices. Prove that

$$\langle m, n, \ell \rangle \cong \mathrm{Id}_M \otimes \mathrm{Id}_N \otimes \mathrm{Id}_L$$

Solution 1 (Typeset by all students in the lecture). Our first observation is the following fact

$$A \otimes B \otimes C \cong (M \otimes N^*) \otimes (N \otimes L^*) \otimes (L \otimes M^*) \cong (M \otimes M^*) \otimes (N \otimes N^*) \otimes (L \otimes L^*) \quad (0.1)$$

We see $X \otimes X^*$ as the space of $(x \times x)$ matrices where x is the dimension of X. Furthermore we can see $\langle m, n, \ell \rangle$ as a trilinear map that lives in $A^* \times B^* \times C^* \to \mathbb{F}$.

Now since $T := \langle m, n, \ell \rangle$ denotes the matrix multiplication tensor we can conclude that for an element $(x_{i,i'}, y_{j,j'}, z_{h,h'}) \in A^* \times B^* \times C^*$ we have :

$$T(x_{i,i'}, y_{j,j'}, z_{h,h'}) = \begin{cases} 1 & \text{if } i' = j, j' = h \text{ and } i = h' \\ 0 & \text{else} \end{cases} \quad (0.2)$$

Using the canonical isomorphism that we get from our first observation and the conditions we get from (0.2) we can see T as the trilinear map

$$T : (M^* \otimes M) \times (N^* \otimes N) \times (L^* \otimes L) \to \mathbb{F}, (x_{h',i}, y_{i',j}, z_{j',h}) \mapsto \begin{cases} 1 & \text{if } h' = i, i' = j \text{ and } j' = h \\ 0 & \text{else} \end{cases}$$

Thus only elements of the form

$$(x_{i,i}, y_{j,j}, z_{h,h})$$

get mapped to 1 and everything else becomes 0.

Now that is exactly the map that corresponds to the tensor

$$\mathrm{Id}_M \otimes \mathrm{Id}_M \otimes \mathrm{Id}_L$$

Exercise 2 (15 Points)**.** Let V be an irreducible GL_n-representation. Prove that V is also irreducible as an SL_n-representation, where SL_n is the group of matrices with determinant equal to 1.

Solution 2 (Typeset by all students in the lecture)**.** Let V be an irreducible GL_n-representation. Now assume that V is not irreducible as an SL_n-representation. That means that there exists a non-trivial linear subspace $W \subseteq V$ such that W is closed under the action of SL_n :

$$\forall s \in \mathsf{SL}_n : \forall w \in W : sw \in W \tag{0.3}$$

Since V is irreducible as a GL_n-representation there is a $g \in \mathsf{GL}_n$ with $\det(g) \neq 1$ (due to (3)) and $w^* \in W$ such that:

$$gw^* \notin W \tag{0.4}$$

Now define :

$$s := \frac{1}{\sqrt[n]{\det(g))}} g$$

Note that $\det(s) = 1$ such that $s \in \mathsf{SL}_n$.

Using (0.3) we further get:

$$sw^* \in W$$

But since W was a linear subgroup of V we also have that

$$\sqrt[n]{\det(g)} \cdot sw^* = \sqrt[n]{\det(g)} \cdot \frac{1}{\sqrt[n]{\det(g))}} gw^* = gw^* \in W$$

which contradicts (0.4).

Thus W is irreducible as an SL_n-representation.

Exercise 3 (10 Points)**.** Let W be a vector space of dimension 2 and let $\ell, \ell_1, \ell_2, \dots, \ell_{n-1} \in W$ be pairwise independent, i.e., $\dim(\langle a, b\rangle) = 2$ for all $a, b \in \{\ell, \ell_1, \ell_2, \dots, \ell_{n-1}\}$ with $a \neq b$. Prove that there is a $g \in S^{n\text{-}1}W^*$ that vanishes on $\ell_1, \ell_2, \dots, \ell_{n-1}$, but not on ℓ.

Solution 3. Since $\dim(\langle \ell, \ell_1\rangle) = 2$, we can assume that $\{\ell, \ell_1\}$ is a basis of W. Thus there exist scalars $\lambda_2, \mu_2, \lambda_3, \mu_3, \dots, \lambda_{n-1}, \mu_{n-1}$ such that $\ell_j = \lambda_j \ell + \mu_j \ell_1$ for $j \in \{2, 3, \dots, n-1\}$. Note that $\lambda_2, \mu_2, \lambda_3, \mu_3, \dots, \lambda_{n-1}, \mu_{n-1}$ are all non-zero because otherwise some ℓ_j (for $j > 2$) would not be independent of ℓ or ℓ_1. Now let $x, y \in W^*$ be such that $x(\ell) = 1, x(\ell_1) = 0, y(\ell) = 0, y(\ell_1) = 1$.

Thus $\{x, y\}$ is a basis of W^*. Now consider the following elements $f_1, f_2, \dots, f_{n-1}$ of W^*, where $f_1 = x$ and $f_j = \mu_j x - \lambda_j y$ for $j \in \{2, 3, \dots, n-1\}$. It is easy to observe that $f_j(\ell_j) = 0$ for $j \in \{1, 2, \dots, n-1\}$. Consider now $g = f_1 \cdot f_2 \cdots \cdot f_{n-1}$. Note that f is a bi-variate homogeneous polynomial in x, y of degree $n-1$. Thus $g \in S^{n\text{-}1}W^*$, and by the choice of f_j's we know that g vanishes on $\ell_1, \ell_2, \dots, \ell_{n-1}$. Also $g(\ell) = 1 \cdot \mu_2 \cdot \mu_3 \cdot \cdots \cdot \mu_{n-1}$, since all μ_j's are non-zero we get that $g(\ell) \neq 0$. Thus g vanishes on $\ell_1, \ell_2, \dots, \ell_{n-1}$, but not on ℓ.