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Exercises for Geometric complexity theory 2

https://people.mpi-inf.mpg.de/~cikenmey/teaching/winter1718/gct2/index.html

Exercise sheet 6 Solutions   Due: **Tuesday, December 12, 2017**

Total points : 40

Exercise 1 (10 Points)**.** Prove that the unit tensor $\sum_{i=1}^n e_i \otimes e_i \otimes e_i \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ is characterized by its stabilizer (which is a subgroup in $\mathsf{GL}_n \times \mathsf{GL}_n \times \mathsf{GL}_n$). You do not have to determine the stabilizer.

Solution 1. We use the notation $D(a_1, a_2, \ldots, a_n)$ to denote the $n \times n$ diagonal matrix with $a_1, \ldots, a_n$ on the diagonal. Let ζ be a primitive third root of unity. Consider the following subgroup H of $\mathsf{GL}_n \times \mathsf{GL}_n \times \mathsf{GL}_n$.

$$H = \{(h,h,h) \mid h = D(\zeta^{\ell_1}, \zeta^{\ell_2}, \ldots, \zeta^{\ell_n}), \ell_i \in \{0,1,2\}\} \simeq (\mathbb{Z}_3)^n$$

It is easy to see that $\sum_{i=1}^n e_i \otimes e_i \otimes e_i$ is invariant under the action of H. Also, let $e_i \otimes e_j \otimes e_k$ be a basis vector of $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$, where i, j, k are not equal to each other, i.e, $e_i \otimes e_j \otimes e_k \neq e_i \otimes e_i \otimes e_i$. We show now that under the symmetrization of H, $e_i \otimes e_j \otimes e_k$ maps to zero. WLOG assume that $k \neq j$ and $k \neq i$ (but we allow $i = j$ or $i \neq j$). We consider $G_k \leq H$, $G_k := \{(h,h,h) \mid h = D(1,1,\ldots,1,\zeta^\ell,1,\ldots,1), \ell \in \{0,1,2\}\} \simeq \mathbb{Z}_3$, where ζ^ℓ is at position k. Thus we have

$$\begin{aligned} \sum_{(h,h,h)\in G_k} he_i \otimes he_j \otimes he_k &= \sum_{k'\in\{0,1,2\}} e_i \otimes e_j \otimes \zeta^{k'} e_k \\ &= (1+\zeta+\zeta^2) e_i \otimes e_j \otimes e_k \\ &= 0 \end{aligned}$$

Thus if a tensor $v \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ is invariant under the action of H then only rank-1 tensors of the form $e_i \otimes e_i \otimes e_i$ can appear in the tensor decomposition of v. Thus we can assume that $v = \sum_{i=1}^n \alpha_i e_i \otimes e_i \otimes e_i$ with $\alpha_i \in \mathbb{C}$.

Now we further look at the action of the subgroup $S = ((\sigma,\sigma,\sigma) \mid \sigma \in \mathfrak{S}_n)$. It is clear that if v is invariant under the action of S then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$. Thus $v = \alpha \sum_{i=1}^n e_i \otimes e_i \otimes e_i$. Hence $\sum_{i=1}^n e_i \otimes e_i \otimes e_i$ is characterized by its stabilizer.

Following is an alternative solution typeset by all students in the lecture.

We begin the characterization of $v = \sum_{i=1}^n e_i \otimes e_i \otimes e_i$ by first looking at the following group elements:

$$S_1 := \left\{ (I_n + (\alpha - 1)E_{j,j}, I_n + (\alpha - 1)E_{j,j}, I_n + (\frac{1}{\alpha^2} - 1)E_{j,j}) \mid j \in \mathbb{N}, \alpha \in \mathbb{C} \setminus \{0\} \right\}$$

where $E_{i,j} \in \mathsf{GL}_n$ is the matrix with a 1 in row i and column j and zeroes everywhere else. S_1 is basically the set of all group elements that rescale one of the basis elements and don't change the others, where the rescaling is α in the first two components and $\frac{1}{\alpha^2}$ in the last component. This set is in the stabilizer of v: For any $g \in S_1$, there is only one vector $e_{i_0} \otimes e_{i_0} \otimes e_{i_0}$ that is affected by the rescaling of e_{i_0}. In this vector, the scalings obviously cancel each other out.

S_1 also gives us the first restriction: Any other vector with S_1 in its stabilizer has the form $\sum_{i=1}^n \beta_i e_i \otimes e_i \otimes e_i$. Let $v = \sum_{1 \le i,j,k \le n} \beta_{i,j,k} e_i \otimes e_j \otimes e_k$ be a vector written as its decomposition into the standard basis. Any $e_i \otimes e_j \otimes e_k$ with i, j, k not all equal, will change its scalar for a $g \in S_1$ that rescales either e_i, e_j or e_k. This means that any vector with a $\beta_{i,j,k} \neq 0$ for i, j, k not all equal, will not have the whole S_1 in its stabilizer.

Knowing that only vectors of the form $w = \sum_{i=1}^n \beta_i e_i \otimes e_i \otimes e_i$ might have the same stabilizer as v allows us to continue the characterization by using the symmetric group. The action $\rho(\sigma) = e_i \mapsto e_{\sigma(i)}$ can be embedded into the GL_n. We now look how a triple of transpositions $\sigma_{i,j} = ((ij), (ij), (ij))$ changes $\sum_{i=1}^n \beta_i e_i \otimes e_i \otimes e_i$: since the e_i get swapped with the e_j, we now know that after the action of $\sigma_{i,j}$ the coefficients β_i and β_j have switched places. Thus we know that a vector w defined as above has $\sigma_{i,j}$ in its stabilizer if and only if $\beta_i = \beta_j$.

With this argument, we now know that S_1 and $S_2 := \{\sigma_{i,j} \mid 1 \le i, j \le n, i \neq j\}$ are in the stabilizer of v and uniquely characterize v.

Exercise 2 (20 Points)**.** Let $\lambda \vdash n$. We know from Gay's theorem that the $(n \times 1)$-weight space in the irreducible GL_n representation $\{\lambda\}$ is isomorphic to the Specht module $[\lambda]$ as an $\mathfrak{S}_n$-representation.

For some n of your choice, find a partition $\mu \vdash 2n$ such that the $(n \times 2)$-weight space of $\{\mu\}$ is not irreducible as an $\mathfrak{S}_n$-representation.

Solution 2. We choose $n = 3$ and $\mu = (4, 2)$. The (3×2)-weight space $\{\nu\}$ of $\{\mu\}$ is spanned by the following tableaux.

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&3\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 3&3\\\cline{1-2}\end{array} \text{ and } \begin{array}{|c|c|c|c|}\hline 1&1&3&3\\\hline 2&2\\\cline{1-2}\end{array}$$

Thus $\dim(\{\nu\}) = 3$. But then $\{\nu\}$ can not be an irreducible $\mathfrak{S}_3$-representation because dimension of any irreducible $\mathfrak{S}_3$-representation ie either 1 or 2. This is because $[3], [2, 1]$ and $[1, 1, 1]$ are the only irreducible $\mathfrak{S}_3$-representations, we have $\dim([1, 1, 1]) = \dim([3]) = 1$ and $\dim([2, 1]) = 2$.

Exercise 3 (10 Points)**.** Let $v := x_1^3 + x_2^3 \in \operatorname{Sym}^3 \mathbb{C}^2$. Determine the multiplicities

$$\operatorname{mult}_{(5,4)} \mathbb{C}[\mathsf{GL}_2 v]_3$$

and

$$\operatorname{mult}_{(6,3)} \mathbb{C}[\mathsf{GL}_2 v]_3.$$

Solution 3 (Typeset by all students in the lecture)**.** We know $\mathrm{mult}_{(5,4)}\mathbb{C}[\mathsf{GL}_2 v]_3 = \dim\left(\{\lambda\}^{\mathrm{Stab}(v)}\right)$. As characterized in the lecture $\mathrm{Stab}(v) = \mathbb{Z}_3^2 \rtimes \mathfrak{S}_2$ and a basis of $\{\lambda\}^{\mathbb{Z}_3^2}$ is given by Young tableaux of shape λ where the numbers 1 and 2 appear a multiple of 3 times.

For $\lambda = (5,4)$ we have to start with $\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & & \\ \hline 2 & 2 & 2 & \\ \cline{1-4}\end{array}$ but can not fill it such that numbers 1 and 2 appear a multiple of 3 times. So $\{(5,4)\}^{\mathbb{Z}_3^2} = 0$ and thus $\mathrm{mult}_{(5,4)}\mathbb{C}[\mathsf{GL}_2 v]_3$ is already 0.

For $\lambda = (6,3)$ we only have two non-zero valid generating Young tableaux in $\{(6,3)\}^{\mathbb{Z}_3^2}$:

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3}\end{array} \text{ and } \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 & 2 \\ \hline 2 & 2 & 2 \\ \cline{1-3}\end{array}$$

Symmetrizing those over $\mathfrak{S}_2$ both yield

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3}\end{array} - \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 & 2 \\ \hline 2 & 2 & 2 \\ \cline{1-3}\end{array}$$

Thus $\mathrm{mult}_{(6,3)}\mathbb{C}[\mathsf{GL}_2 v]_3 = 1$.