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Exercises for Geometric complexity theory 2

https://people.mpi-inf.mpg.de/~cikenmey/teaching/winter1718/gct2/index.html

Exercise sheet 7 Solutions Due: **Tuesday, December 19, 2017**

Total points : 40

Let T_{SL_n} denote the group of diagonal matrices with determinant 1. Consider the group $G := T_{\mathsf{SL}_n} \times T_{\mathsf{SL}_n}$, which acts on $\mathrm{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n)$ and preserves both the determinant and the permanent. The group $\mathfrak{S}_n \otimes \mathfrak{S}_n$ acts on the space of G-invariants $V := (\mathrm{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n))^G$ and we saw in the lecture that per_n is the unique polynomial (up to scale) of type $((n),(n))$ in V, whereas $\det_n$ is the unique polynomial (up to scale) of type $((1^n),(1^n))$ in V. In the following exercises, consider $W := (\mathrm{Sym}^{2n}(\mathbb{C}^n \otimes \mathbb{C}^n))^G$. Analogously to V, there is a canonical action of $\mathfrak{S}_n \otimes \mathfrak{S}_n$ on W.

By a result in the lecture we know that if $|\lambda| = |\mu|$ then

$$\dim\left(([\lambda] \otimes [\mu])^{\mathfrak{S}_{|\lambda|}}\right) = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases} \tag{0.1}$$

By Schur-Weyl duality we know that

$$\mathrm{Sym}^{2n}(\mathbb{C}^n \otimes \mathbb{C}^n) \cong \bigoplus_{\lambda,\mu \vdash_n 2n} \{\lambda\} \otimes \{\mu\} \otimes ([\lambda] \otimes [\mu])^{\mathfrak{S}_{2n}}$$

Thus by using (0.1) we get that

$$\mathrm{Sym}^{2n}(\mathbb{C}^n \otimes \mathbb{C}^n) \cong \bigoplus_{\lambda \vdash_n 2n} \{\lambda\} \otimes \{\lambda\}$$

Hence

$$W = \left(\mathrm{Sym}^{2n}(\mathbb{C}^n \otimes \mathbb{C}^n)\right)^{T_{\mathsf{SL}_n} \times T_{\mathsf{SL}_n}} \cong \bigoplus_{\lambda \vdash_n 2n} \{\lambda\}^{T_{\mathsf{SL}_n}} \otimes \{\lambda\}^{T_{\mathsf{SL}_n}}$$

If $\lambda \vdash_n 2n$ then $\{\lambda\}^{T_{\mathsf{SL}_n}}$ has a basis of semi-standard tableaux of shape λ in which each number $k \in \{1, 2, \ldots, n\}$ appears same number of times, i.e, exactly 2 times.

Thus we have the following lemma.

Lemma 1. *If* $\lambda \vdash_n 2n$ and

$$V_\lambda := \text{Vector space spanned by semi-standard tableaux of shape } \lambda$$
$$\text{in which each number from } \{1,2,\dots,n\} \text{ appears exactly twice.}$$

Then

$$W \cong \bigoplus_{\lambda \vdash_n 2n} V_\lambda \otimes V_\lambda$$

Exercise 1 (4 Points)**.** Prove that in W there are nonzero polynomials of type $((n),(n))$.

Solution 1. By lemma 1, we know that $\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&2&2&\dots&\dots&n&n\\\hline\end{array} \otimes \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&2&2&\dots&\dots&n&n\\\hline\end{array}$ is an element of W and is of type $((n),(n))$ because because both the components are invariant under the action of $\mathfrak{S}_n \otimes \mathfrak{S}_n$.

Exercise 2 (4 Points)**.** Let $n > 1$ and prove that in W the polynomials of type $((n),(n))$ are not unique up to scale.

Solution 2. By lemma 1, we know that $\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline \dots&\dots\\\hline \dots&\dots\\\hline n&n\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline \dots&\dots\\\hline \dots&\dots\\\hline n&n\\\hline\end{array}$ is an element of W and is of type $((n),(n))$ because both the components are invariant under the action of $\mathfrak{S}_n \otimes \mathfrak{S}_n$.

Exercise 3 (4 Points)**.** Prove that in W there are nonzero polynomials of type $((1^n),(1^n))$.

Solution 3. By lemma 1, we know that $\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&\dots&n\\\hline 2\\\cline{1-1} 3\\\cline{1-1} \dots\\\cline{1-1} \dots\\\cline{1-1} n\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&\dots&n\\\hline 2\\\cline{1-1} 3\\\cline{1-1} \dots\\\cline{1-1} \dots\\\cline{1-1} n\\\cline{1-1}\end{array}$ is an element of W. Also, it is of type $((1^n),(1^n))$ because we have the following equation for action of any $(\sigma,\pi) \in \mathfrak{S}_n \otimes \mathfrak{S}_n$.

$$(\sigma,\pi)\cdot \begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&\dots&n\\\hline 2\\\cline{1-1} 3\\\cline{1-1} \dots\\\cline{1-1} \dots\\\cline{1-1} n\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&\dots&n\\\hline 2\\\cline{1-1} 3\\\cline{1-1} \dots\\\cline{1-1} \dots\\\cline{1-1} n\\\cline{1-1}\end{array} = \operatorname{sgn}(\sigma)\cdot \begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&\dots&n\\\hline 2\\\cline{1-1} 3\\\cline{1-1} \dots\\\cline{1-1} \dots\\\cline{1-1} n\\\cline{1-1}\end{array} \otimes \operatorname{sgn}(\pi)\cdot \begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&\dots&n\\\hline 2\\\cline{1-1} 3\\\cline{1-1} \dots\\\cline{1-1} \dots\\\cline{1-1} n\\\cline{1-1}\end{array}$$

Exercise 4 (12 Points)**.** Let $n = 3$ and prove that in W the polynomials of type $((1^3),(1^3))$ are not unique up to scale.

Solution 4. It is easy to verify that $\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}$ and $\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3&3\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3&3\\\hline\end{array}$ are of type $((1^3),(1^3))$.

Exercise 5 (16 Points)**.** Prove that for $n = 6$ the space W contains a nonzero polynomial of type $((6), (1^6))$.

You can use that the GL_6 -representation $\mathrm{Sym}^6(\mathrm{Sym}^2 \mathbb{C}^6)$ decomposes as $\{(12)\} \oplus \{(10,2)\} \oplus \{(8,4)\} \oplus \{(8,2^2)\} \oplus \{(6^2)\} \oplus \{(6,4,2)\} \oplus \{(6,2^3)\} \oplus \{(4^3)\} \oplus \{(4^2,2^2)\} \oplus \{(4,2^4)\} \oplus \{(2^6)\}$

and that the GL_6 -representation $\bigwedge^6(\mathrm{Sym}^2 \mathbb{C}^6)$ decomposes as $\{(7,1^5)\} \oplus \{(6,3,1^3)\} \oplus \{(5,4,2,1)\} \oplus \{(4^3)\}$.

Solution 5. We use the following fact which easily follows from Schur-Weyl duality : $\mathrm{HWV}_{(4^3)}(\otimes^6(\mathrm{Sym}^2 \mathbb{C}^6) \cong [4^3]^{\mathfrak{S}_2^6}$ has a basis consisting of semi-standard tableaux of shape (4^3) in which each entry appears twice (see Corollary 18.3.11 in `http://people.mpi-inf.mpg.de/~cikenmey/teaching/summer17/introtogct/gct.pdf`).

By Proposition 18.3.2 in `http://people.mpi-inf.mpg.de/~cikenmey/teaching/summer17/introtogct/gct.pdf`, we know that

$$a_{(4^3)}(6,2) = \dim[4^3]^{\mathfrak{S}_2 \wr \mathfrak{S}_6} = \dim(\mathrm{HWV}_{(4^3)}(\mathrm{Sym}^6(\mathrm{Sym}^2 \mathbb{C}^6))$$

Since we have

$$\mathrm{Sym}^6(\mathrm{Sym}^2 \mathbb{C}^6) \cong \{(12)\} \oplus \{(10,2)\} \oplus \{(8,4)\} \oplus \{(8,2^2)\} \oplus \{(6^2)\} \oplus \\ \{(6,4,2)\} \oplus \{(6,2^3)\} \oplus \{(4^3)\} \oplus \{(4^2,2^2)\} \oplus \{(4,2^4)\} \oplus \{(2^6)\}$$

We get that $a_{(4^3)}(6,2) \neq 0$. On the other hand, we know that elements of $[4^3]^{\mathfrak{S}_2 \wr \mathfrak{S}_6}$ are linear combinations of semi-standard tableaux of shape (4^3) and of content 6×2, which are invariant under the action of $\mathfrak{S}_6$. Thus elements of $[4^3]^{\mathfrak{S}_2 \wr \mathfrak{S}_6}$ are of type (6).

We know that $[4^3]^{\mathfrak{S}_2^6}$ is spanned by semi-standard tableaux of shape (4^3) and of content 6×2. Let us define

$$U := \{T \in [4^3]^{\mathfrak{S}_2^6} \mid \forall \sigma \in \mathfrak{S}_6 : \sigma T = \mathrm{sgn}(\sigma) T\}$$

Note that the elements of U are of the type (1^6).

We have that $\dim(\mathrm{HWV}_{(4^3)}(\bigwedge^6(\mathrm{Sym}^2 \mathbb{C}^6)) = \dim(U)$. Since we have

$$\bigwedge^6(\mathrm{Sym}^2 \mathbb{C}^6) \cong \{(7,1^5)\} \oplus \{(6,3,1^3)\} \oplus \{(5,4,2,1)\} \oplus \{(4^3)\}$$

We get that $\dim(U) \neq 0$.

Let $w_1 \in [4^3]^{\mathfrak{S}_2 \wr \mathfrak{S}_6}$ and $w_2 \in U$ be any non-zero elements. Then $w_1 \otimes w_2$ is a nonzero polynomial of type $((6), (1^6))$ in $W = \left(\mathrm{Sym}^{12}(\mathbb{C}^6 \otimes \mathbb{C}^6)\right)^{T_{\mathsf{SL}_6} \times T_{\mathsf{SL}_6}}$.