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Exercises for Geometric complexity theory 2

https://people.mpi-inf.mpg.de/~cikenmey/teaching/winter1718/gct2/index.html

Exercise sheet 8 Solutions Due: **Tuesday, January 2, 2018**

Total points : 40

Exercise 1 (10 Points)**.** Prove that every complex matrix of finite order is diagonalizable.

Solution 1. The minimal polynomial μ_A of an $n \times n$ matrix A is the monic polynomial P of least degree such that $P(A) = 0$. Any other polynomial Q with $Q(A) = 0$ is a (polynomial) multiple of μ_A. We shall use the following lemma for diagonalizable matrices.

Lemma 1. *A matrix A is diagonalizable if and only if its minimal polynomial μ_A factors completely into distinct linear factors.*

For an example where μ_M does not satisfy the condition of lemma 1, look at the following matrix $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and note that $\mu_M = (x-1)^3$.

Suppose $A \in \mathbb{C}^{n \times n}$ has finite order, thus there exists $k \geq 1$ such that $A^k = I_n$. Thus for $Q(x) = x^k - 1$, we have $Q(A) = 0$. Thus $\mu_A \mid Q(x)$. Since $Q(x)$ factors completely into distinct linear factors, so does μ_A. Thus A is diagonalizable.

Exercise 2 (10 Points)**.** Fix a natural number n. Given a list $(c_1, c_2, \ldots, c_\ell) \in \{1, 2, \ldots, n\}^\ell$ of pairwise distinct numbers, the corresponding cycle is the permutation π that satisfies $\pi(c_i) = c_{i+1}$ for all $1 \leq i \leq \ell - 1$, $\pi(c_\ell) = c_1$, and $\pi(j) = j$ if $\forall i : j \neq c_i$. The number ℓ is called the length of the cycle and the set $\{c_1, c_2, \ldots, c_\ell\}$ is called its support. Two cycles are called disjoint if their supports have empty intersection. Clearly disjoint cycles commute. Prove that every $\pi \in \mathfrak{S}_n$ can be written uniquely (up to a permutation of the factors) as a product of disjoint cycles. This is called the cycle decomposition.

Solution 2. Let $\pi \in \mathfrak{S}_n$ be any permutation. Pick $a \in \{1, 2, \ldots, n\}$ and consider the sequence $a, \pi(a), \pi^2(a), \ldots$. This sequence must eventually repeat, so there exist $k < \ell$ such that $\pi^k(a) = \pi^\ell(a)$.Thus we have $\pi^{\ell-k}(a) = a$. Let r be the smallest positive integer such that $\pi^r(a) = a$. Then, we have a cycle:

$$(a \longrightarrow \pi(a) \longrightarrow \pi^2(a) \longrightarrow \ldots \longrightarrow \pi^{r-1}(a) \longrightarrow a).$$

Now repeat the same process for every element $b \in \{1, 2, \dots, n\} \setminus \{a, \pi(a), \pi^2(a), \dots, \pi^{r-1}(a\}$, to get the cycle decomposition of π.

Exercise 3 (10 Points)**.** Let $\pi = c^{(1)}c^{(2)} \dots c^{(k)} \in \mathfrak{S}_n$ be the cycle decomposition. Let ℓ_i denote the length of the cycle $c^{(i)}$. If we sort the list of lengths $(\ell_1, \ell_2, \dots, \ell_k)$, then we obtain a partition $\lambda \vdash n$ that we call the cycle type of π. Prove that two permutations are in the same conjugacy class iff they have the same cycle type.

Solution 3. Suppose σ and π are conjugates, i.e., there exists $g \in \mathfrak{S}_n$ such that $\sigma = g\pi g^{-1}$. To prove that σ and π have same cycle type, it is enough to show that $\pi(i) = j \iff \sigma(g(i)) = g(j)$. Suppose $\pi(i) = j$, then we have $\sigma(g(i)) = g\pi g^{-1}(g(i)) = g(\pi(i)) = g(j)$. The other direction follows similarly.

Now suppose σ and π have the same cycle type. Write the cycle decomposition for each permutation in such a way that the cycles are listed in non-decreasing order of their length (including cycles of length 1). We then have (for example)

$$\begin{aligned} \sigma &= (a_1)(a_2)(a_3a_4)(a_5a_6a_7)\dots(a_{10}\dots a_n) \\ \pi &= (b_1)(b_2)(b_3b_4)(b_5b_6b_7)\dots(b_{10}\dots b_n) \end{aligned}$$

Define $g \in \mathfrak{S}_n$ to be the permutation that takes a_i to b_i. Clearly $g \in \mathfrak{S}_n$, since each of $1, \dots, n$ appears exactly once among the a_i and once among the b_i. Let a_j, b_j denote the "next" elements to a_i, b_i in their respective cycles σ and π. Then we have

$$g\sigma g^{-1}(b_i) = g\sigma(a_i) = g(a_j) = b_j = \pi(b_i).$$

Thus $\pi = g\sigma g^{-1}$.

Exercise 4 (10 Points)**.** Let G be a finite group and let V and W be two G-representations (in particular, V and W are finite dimensional). Then the direct sum $V \oplus W$ of vector spaces is a G-representation via

$$g(v, w) := (gv, gw).$$

Prove that the character $\chi_{V\oplus W}$ satisfies $\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g)$.

Solution 4. Let $n = \dim(V)$ and $m = \dim(W)$. For $g \in G$, let $g_1 \in \mathsf{GL}_n$ and $g_2 \in \mathsf{GL}_m$ be the corresponding matrices, i.e, $\forall v \in V : g_1 v = gv$ and $\forall w \in W : g_2 w = gw$. Similarly, let $g' \in \mathsf{GL}_{n+m}$ be corresponding matrix, i.e, $\forall (v, w) \in V \oplus W : g'(v, w) = g(v, w)$. Now observe that $g' = g_1 \oplus g_2$, i.e., we have

$$g' = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}.$$

Thus we have

$$\chi_{V\oplus W}(g) = \mathrm{Tr}(g') = \mathrm{Tr}(g_1) + \mathrm{Tr}(g_2) = \chi_V(g) + \chi_W(g).$$