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## Exercises for Geometric complexity theory 2

<https://people.mpi-inf.mpg.de/~cikenmey/teaching/winter1718/gct2/index.html>

Exercise sheet 9 Solutions

Due: **Tuesday, January 9, 2018**

Total points : 40

**Exercise 1** (10 Points). Let  $G$  be a finite group and let  $V$  and  $W$  be two  $G$ -representations (in particular,  $V$  and  $W$  are finite dimensional). Then the tensor product  $V \otimes W$  of vector spaces is a  $G$ -representation via

$$g(v \otimes w) := gv \otimes gw$$

and linear continuation. Prove that the character  $\chi_{V \otimes W}$  satisfies  $\chi_{V \otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$ .

**Solution 1.** Let  $n = \dim(V)$  and  $m = \dim(W)$ . Let  $v_1, v_2, \dots, v_n$  and  $w_1, w_2, \dots, w_m$  be some basis of  $V$  and  $W$  respectively. For  $g \in G$ , let  $A \in \mathbf{GL}_n$  and  $B \in \mathbf{GL}_m$  be the corresponding matrices, i.e.,  $\forall v \in V : Av = gv$  and  $\forall w \in W : Bw = gw$ . These matrices  $A, B$  correspond to the basis  $v_1, v_2, \dots, v_n$  and  $w_1, w_2, \dots, w_m$ .

Similarly, let  $C \in \mathbf{GL}_{nm}$  be the corresponding matrix, i.e.,  $\forall t \in V \otimes W : C(t) = g(t)$ . This matrix corresponds to the basis  $v_i \otimes w_j$  of  $V \otimes W$ . We have  $Av_i = \sum_{k=1}^n A_{ik}v_k$ ,  $Bw_j = \sum_{\ell=1}^m B_{j\ell}w_\ell$  and

$$C(v_i \otimes w_j) = \left( \sum_{k=1}^n A_{ik}v_k \right) \otimes \left( \sum_{\ell=1}^m B_{j\ell}w_\ell \right) = \sum_{k=1}^n \sum_{\ell=1}^m A_{ik}B_{j\ell}v_k \otimes w_\ell \quad (0.1)$$

The coefficient of  $v_i \otimes w_j$  in equation (0.1) is  $A_{ii}B_{jj}$ . Thus

$$\mathrm{tr}(C) = \chi_{V \otimes W}(g) = \sum_{i=1}^n \sum_{j=1}^m A_{ii}B_{jj} = \left( \sum_{i=1}^n A_{ii} \right) \cdot \left( \sum_{j=1}^m B_{jj} \right) = \mathrm{tr}(A) \cdot \mathrm{tr}(B) = \chi_V(g) \cdot \chi_W(g).$$

One can also see that  $C = A \otimes B$ .

**Exercise 2** (10 Points). The tensor product of Specht modules  $[(2,1)] \otimes [(2,1)]$  is a 4-dimensional  $\mathfrak{S}_3$ -representation. Compute its character and decompose it as a linear combination of characters of irreducible  $\mathfrak{S}_3$ -representations.

**Solution 2.** The character is constant on conjugacy classes, so we only have to consider one group element from each conjugacy class. The conjugacy classes were determined on the last exercise sheet.

The character table of  $\mathfrak{S}_3$  looks like below.

	$e$	$(1, 2)$	$(1, 2, 3)$
$\chi_{[(3)]}$	1	1	1
$\chi_{[(1,1,1)]}$	1	-1	1
$\chi_{[(2,1)]}$	2	0	-1

Thus by using exercise 1, we can describe the characters of  $[(2, 1)] \otimes [(2, 1)]$  as below.

	$e$	$(1, 2)$	$(1, 2, 3)$
$\chi_{[(2,1)] \otimes [(2,1)]}$	4	0	1

Now it is easy to verify that  $\chi_{[(2,1)] \otimes [(2,1)]} = \chi_{[(3)]} + \chi_{[(1,1,1)]} + \chi_{[(2,1)]}$ .

**Exercise 3** (10 Points). The tensor power of Specht modules  $W := [(2, 1)]^{\otimes n}$  is a  $2^n$ -dimensional  $\mathfrak{S}_3$ -representation via

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_n$$

and linear continuation. Determine the multiplicities of the irreducible  $\mathfrak{S}_3$ -representations in  $W$ .

**Solution 3.** By using a similar argument as in exercise 2, we can describe the characters of  $W := [(2, 1)]^{\otimes n}$  as below.

	$e$	$(1, 2)$	$(1, 2, 3)$
$\chi_{[(2,1)]^{\otimes n}}$	$2^n$	0	$(-1)^n$

Suppose we have that  $\text{mult}_{[(3)]}([(2, 1)]^{\otimes n}) = x$ ,  $\text{mult}_{[(1,1,1)]}([(2, 1)]^{\otimes n}) = y$  and  $\text{mult}_{[(2,1)]}([(2, 1)]^{\otimes n}) = z$ . By using the last exercise in Assignment 8, we know that

$$\chi_{[(2,1)]^{\otimes n}} = x \cdot \chi_{[(3)]} + y \cdot \chi_{[(1,1,1)]} + z \cdot \chi_{[(2,1)]}$$

Solving for  $x, y, z$ , we obtain that  $x = y = \frac{2^{n-1} + (-1)^n}{3}$  and  $z = \frac{2^n - (-1)^n}{3}$ .

**Exercise 4** (10 Points). Let  $H \leq G$  be a subgroup of a (not necessarily finite) group. For each  $g \in G$  we define the *coset* as its orbit under the right multiplication:

$$gH := \{gh \mid h \in H\}.$$

Prove that distinct cosets have empty intersection. Also prove that all cosets have the same cardinality.

**Solution 4.** Let  $g_1H$  and  $g_2H$  be two cosets. Assume that  $g_1H \cap g_2H \neq \emptyset$ . Let  $g \in g_1H \cap g_2H$ , thus  $g = g_1h_1 = g_2h_2$  for some  $h_1, h_2 \in H$ . This implies that  $g_1 = g_2h_2h_1^{-1}$ . Now we show that  $g_1H = g_2H$ . Let  $g_1h$  be some arbitrary element of  $g_1H$ . We have

$$g_1h = g_2h_2h_1^{-1}h.$$

Thus  $g_1h \in g_2H$ , giving us that  $g_1H \subseteq g_2H$ . Similarly we can show that  $g_2H \subseteq g_1H$ . Hence  $g_1H = g_2H$ .

The map  $H \rightarrow gH, h \mapsto gh$ , is a bijection: It's inverse map is given by  $gH \rightarrow H, j \mapsto g^{-1}j$ . Thus  $H$  and  $gH$  have the same cardinality and hence *all* cosets have the same cardinality.