Prof. Dr. Markus Bläser and Dr. Christian Ikenmeyer **Winter 2017/2018**

Exercises for Geometric complexity theory 2

https://people.mpi-inf.mpg.de/~cikenmey/teaching/winter1718/gct2/index.html

Exercise sheet 10 Solutions Due: **Tuesday, January 16, 2018**

Total points : 40

For $i, j \in \mathbb{N}$ and a partition $\lambda \vdash_i ij$ let $\{\lambda\}^{i\times j}$ denote the $(i \times j)$-weight space in the irreducible GL_i-representation $\{\lambda\}$. For $\nu \vdash i$ we define the generalized plethysm coefficient $a_\lambda(\nu, j)$ via embedding $\mathfrak{S}_i \hookrightarrow \mathsf{GL}_i$ and decomposing:

$$\{\lambda\}^{i\times j} = \bigoplus_{\nu\vdash i} a_\lambda(\nu, j)[\nu].$$

Exercise 1 (5 Points)**.** Prove that

$$a_\lambda(\nu, 1) = \begin{cases} 1 & \text{if } \lambda = \nu \\ 0 & \text{otherwise} \end{cases}$$

Solution 1. We know that $\{\lambda\}^{i\times 1} = [\lambda]$. Sice $[\lambda]$ is irreducible,

$$a_\lambda(\nu, 1) = \begin{cases} 1 & \text{if } \lambda = \nu \\ 0 & \text{otherwise} \end{cases}$$

Exercise 2 (10 Points)**.** For $i, j \in \mathbb{N}$ the space $\bigotimes^i \operatorname{Sym}^j V$ has a canonical action of $\mathfrak{S}_i \times \mathsf{GL}(V)$. Prove that as an $(\mathfrak{S}_i \times \mathsf{GL}(V))$-representation we have

$$\bigotimes^i \operatorname{Sym}^j V = \bigoplus_{\substack{\lambda\vdash ij\\ \nu\vdash i}} a_\lambda(\nu, j)[\nu] \otimes \{\lambda\}.$$

Solution 2 (Typeset by all students in the lecture)**.** Schur-Weyl duality yields:

$$\bigotimes^i \operatorname{Sym}^j V \cong (\bigotimes^{ij} V)^{\mathfrak{S}_j^i} \cong (\bigoplus_{\lambda\vdash ij} \{\lambda\} \otimes [\lambda])^{\mathfrak{S}_j^i} \cong \bigoplus_{\lambda\vdash ij} \{\lambda\} \otimes [\lambda]^{\mathfrak{S}_j^i}$$

Now using a result from last semester we can write $[\lambda]^{\mathfrak{S}_j^i}$ as $\{\lambda\}^{i\times j}$ which then eventually yields:

$$\begin{aligned}
\bigotimes^i \operatorname{Sym}^j V &\cong \bigoplus_{\lambda \vdash ij} \{\lambda\} \otimes \{\lambda\}^{i \times j} \\
&\cong \bigoplus_{\lambda \vdash ij} \{\lambda\}^{i \times j} \otimes \{\lambda\} \\
&\cong \bigoplus_{\substack{\lambda \vdash ij \\ \nu \vdash i}} a_\lambda(\nu, j)[\nu] \otimes \{\lambda\}
\end{aligned}$$

Exercise 3 (10 Points)**.** For partitions $\nu^1, \nu^2, \dots, \nu^m$ and $\lambda \vdash_i \sum_{i=1}^m |\nu^i|$ let $c^\lambda_{\nu^1,\dots,\nu^m}$ denote the multiplicity of the irreducible $\mathsf{GL}(V)$-representation $\{\lambda\}$ in the tensor product $\{\nu^1\} \otimes \{\nu^2\} \otimes \cdots \otimes \{\nu^m\}$ of irreducible $\mathsf{GL}(V)$-representations. This is called the multi-Littlewood-Richardson coefficient. If $m = 2$, then this is the classical Littlewood-Richardson coefficient. Prove that

$$c^\lambda_{\nu^1,\dots,\nu^m} = \sum_{\substack{\mu^1,\dots,\mu^{m-2} \\ \mu^i \vdash \sum_{j=1}^{i+1} |\nu^j|}} c^{\mu^1}_{\nu^1,\nu^2} \cdot c^{\mu^2}_{\mu^1,\nu^3} \cdot c^{\mu^3}_{\mu^2,\nu^4} \cdot \dots \cdot c^\lambda_{\mu^{m-2},\nu^m}.$$

Solution 3. We prove it by induction on m. For the base case we consider $m = 3$. Thus we have:

$$\{\nu^1\} \otimes \{\nu^2\} \otimes \{\nu^3\} = \left(\bigoplus_{\mu \vdash \nu^1 + \nu^2} \{\mu\}^{\oplus c^\mu_{\nu^1,\nu^2}} \right) \otimes \{\nu^3\}.$$

Hence

$$c^\lambda_{\nu^1,\nu^2,\nu^3} = \sum_{\mu \vdash \nu^1 + \nu^2} c^\mu_{\nu^1,\nu^2} \cdot c^\lambda_{\mu,\nu^3}.$$

For the induction step we have:

$$\{\nu^1\} \otimes \{\nu^2\} \otimes \cdots \otimes \{\nu^m\} = \left(\bigoplus_{\mu \vdash \sum_{i=1}^{m-1} \nu^i} \{\mu\}^{\oplus \sum_{\substack{\mu^1,\dots,\mu^{m-3} \\ \mu^i \vdash \sum_{j=1}^{i+1} |\nu^j|}} c^{\mu^1}_{\nu^1,\nu^2} \cdot c^{\mu^2}_{\mu^1,\nu^3} \cdot c^{\mu^3}_{\mu^2,\nu^4} \cdot \dots \cdot c^{\mu}_{\mu^{m-3},\nu^{m-1}}} \right) \otimes \{\nu^m\}$$

Thus

$$c^\lambda_{\nu^1,\dots,\nu^m} = \sum_{\substack{\mu^1,\dots,\mu^{m-2} \\ \mu^i \vdash \sum_{j=1}^{i+1} |\nu^j|}} c^{\mu^1}_{\nu^1,\nu^2} \cdot c^{\mu^2}_{\mu^1,\nu^3} \cdot c^{\mu^3}_{\mu^2,\nu^4} \cdot \dots \cdot c^\lambda_{\mu^{m-2},\nu^m}.$$

Exercise 4 (15 Points)**.** Embed $\mathfrak{S}_i \times \mathfrak{S}_j \hookrightarrow \mathfrak{S}_{i+j}$ as a Young subgroup (i.e., $\mathfrak{S}_i$ acts on $\{1, 2, \dots, i\}$ and $\mathfrak{S}_j$ acts on $\{i+1, i+2, \dots, i+j\}$). Use Schur-Weyl duality to prove that given a partition $\lambda \vdash i + j$ the irreducible $\mathfrak{S}_{i+j}$-representation $[\lambda]$ decomposes as an $\mathfrak{S}_i \times \mathfrak{S}_j$-representation as follows:

$$[\lambda] = \bigoplus_{\substack{\mu \vdash i \\ \nu \vdash j}} c^{\lambda}_{\mu,\nu}([\mu] \otimes [\nu]).$$

Solution 4 (Typeset by all students in the lecture)**.** Let $V = \mathbb{C}^{i+j}$. Then Schur-Weyl duality (under the action of $\mathfrak{S}_{i+j} \times \mathsf{GL}_{i+j}$) gives us:

$$\bigotimes^{i+j} V \cong \bigoplus_{\lambda \vdash i+j} \{\lambda\} \otimes [\lambda]$$

Now we look at $\bigotimes^{i+j} V \cong (\bigotimes^{i} V) \otimes (\bigotimes^{j} V)$ under the action of $\mathfrak{S}_i \times \mathfrak{S}_j \times \mathsf{GL}_{i+j}$. Using Schur-Weyl duality twice again we finally obtain:

$$\begin{aligned}
\bigotimes^{i+j} V &\cong (\bigotimes^{i} V) \otimes (\bigotimes^{j} V) \\
&\cong (\bigoplus_{\mu \vdash i} \{\mu\} \otimes [\mu]) \otimes (\bigoplus_{\nu \vdash j} \{\nu\} \otimes [\nu]) \\
&\cong \bigoplus_{\substack{\mu \vdash i \\ \nu \vdash j}} (\{\mu\} \otimes \{\nu\}) \otimes ([\mu] \otimes [\nu])
\end{aligned}$$

Therefore we get:

$$\bigoplus_{\lambda \vdash i+j} \{\lambda\} \otimes [\lambda] \cong \bigoplus_{\mu \vdash i, \nu \vdash j} (\{\mu\} \otimes \{\nu\}) \otimes ([\mu] \otimes [\nu]) \tag{0.1}$$

And by comparing both sides of 0.1 and noting that the $\{\lambda\}$ is a GL_{i+j}-representation respectively we obtain:

$$[\lambda] = \bigoplus_{\substack{\mu \vdash i \\ \nu \vdash j}} c^{\lambda}_{\mu,\nu}([\mu] \otimes [\nu])$$