# GCT and symmetries

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In the following lectures we want to study the representation theoretic multiplicities in the coordinate rings of orbits. These give upper bounds for the multiplicities in the coordinate rings of orbit closures. The main tool is the algebraic Peter-Weyl theorem.

## 1 The algebraic Peter-Weyl theorem

Let $\mathbb{A}$ be a finite dimensional complex vector space with a polynomial action of $G = \mathsf{GL}_k$. One main example is $\mathbb{A} = \mathrm{Sym}^n \mathbb{C}^k$. (For tensors $G = \mathsf{GL}_k^3$, $\mathbb{A} = \otimes^3 \mathbb{C}^k$)

Let $Z \subseteq \mathbb{A}$ be locally closed. One main example is a group orbit $Z = Gv$.

Recall (last semester, Chapter 15) that for a locally closed set $Z \subseteq \mathbb{A}$ we defined the coordinate ring $\mathbb{C}[Z]$ as the ring of regular functions (i.e., locally defined by rational functions) on $Z$. Therefore $\mathbb{C}[\overline{Z}] \subseteq \mathbb{C}[Z]$, because $\mathbb{C}[\overline{Z}]$ is the ring of *polynomial* functions on $Z$, and polynomial functions are regular.

For a point $v \in \mathbb{A}$ let $\mathrm{stab}_G(v) \subseteq G$ denote its *stabilizer* (or its *symmetry group*):

$$\mathrm{stab}_G(v) := \{g \in G \mid gv = v\}.$$

The algebraic Peter-Weyl theorem implies that

$$\mathrm{mult}_\lambda(\mathbb{C}[Gv]) = \dim(\{\lambda\}^{\mathrm{stab}_G(v)}).$$

Thus

$$\mathrm{mult}_\lambda(\mathbb{C}[\overline{Gv}]) \leq \dim(\{\lambda\}^{\mathrm{stab}_G(v)}).$$

In our situations more is known by now about the relationship between both rings: One is a so-called *localization* of the other[1].

*This is essentially exactly the same version as from January 29, 2018, with one very minor typo fixed.

[1] Bürgisser and Ikenmeyer, *Fundamental invariants of orbit closures*, Journal of Algebra Volume 477, 1 May 2017, Pages 390–434

# 2 Characterization by the stabilizer

**2.1 Theorem** ([2])**.** *Every connected reductive algebraic subgroup $H \subseteq G$ is characterized (up to group isomorphism) by its dimension data, which is the map $\lambda \mapsto \dim\{\lambda\}^H$.*

**2.2 Definition.** *A point $v \in \mathbb{A}$ is* characterized by its stabilizer *(or alternatively* characterized by its symmetries*) if*

$$\forall w \in \mathbb{A} : \big(\text{stab}_G(v) \subseteq \text{stab}_G(w) \Rightarrow w \in \mathbb{C}v\big).$$

We will see that many points $v$ of interest are characterized by their stabilizer. Since $\text{mult}_\lambda(\mathbb{C}[Gv])$ determines $\text{stab}_G(v)$ up to isomprohism, if in our specific situations a slightly stronger version of Theorem 2.1 holds, then this means that $\text{mult}_\lambda(\mathbb{C}[Gv])$ determines $\text{stab}_G(v)$ and thus $v$.

If something comparable holds for orbit closures (in specific situations), then this would mean that every lower bound can be proved using multiplicity obstructions.

CAVEAT: There are situations in which $\overline{Gw} \not\subseteq \overline{Gv}$ *cannot* be proved using multiplicity obstructions! But in all known cases $v$ and $w$ are not characterized by their stabilizer. For example, let $G$ be the trivial group and let $v \neq w$. Then both $\overline{Gw}$ and $\overline{Gv}$ are just a single point each. But $\mathbb{C}[\overline{Gw}] = \mathbb{C} = \mathbb{C}[\overline{Gv}]$.

# 3 Main Examples

We now determine several stabilizers (or black-box them), prove that the points are characterized by the stabilizer and determine the multiplicities in $\mathbb{C}[Gv]$. On the way we will discuss the basics of the character theory of the symmetric group.

## 3.1 Product of homogeneous linear forms

**3.1 Proposition.** *Let $g \in \mathsf{GL}_k$. If $g(x_1 \cdots x_k) = x_1 \cdots x_k$, then $g$ is the product of a permutation matrix and a diagonal matrix with determinant 1 (the so-called $\mathsf{SL}_k$-torus $T_{\mathsf{SL}_k}$). Notation: $T_{\mathsf{SL}_k} \rtimes \mathfrak{S}_k$ (this is called a semidirect product).*

*Proof.* Clearly every product of a permutation matrix and a diagonal matrix with determinant 1 fixes $x_1 \cdots x_k$.

Let $g(x_1 \cdots x_k) = \ell_1 \cdots \ell_k$. Let $\ell_i = \alpha_{i,1}x_1 + \cdots + \alpha_{i,k}x_k$.

For each $x_{k'}$: Set all other variables to 1. Then $\prod_i \ell_i(1, \ldots, 1, x_{k'}, 1, \ldots, 1) = x_{k'}$. Thus exactly one $\ell_i(1, \ldots, 1, x_{k'}, 1, \ldots, 1)$ is not just a constant, but an affine linear form. This affine linear form is homogeneous, because the polynomial $x_{k'}$ is homogeneous. We conclude that among

[2] Larsen and Pink, *Determining representations from invariant dimensions*, Inventiones Mathematicae, 102:377-389, 1990

the $\ell_i$ at most one can have a nonzero coefficient for the variable $x_{k'}$. On the other hand each $\ell_i$ must have at least one nonzero coefficient, so each $\ell_i = c_i x_{\pi(i)}$ for some permutation $\pi$ and nonzero constants $c_i$. Clearly the constants satisfy $\prod_i c_i = 1$. □

**3.2 Proposition.** *The polynomial $x_1 \cdots x_k$ is characterized by its stabilizer.*

*Proof.* The action of $T_{\mathsf{SL}_k}$ preserves the monomial structure (in other words, the *support* of the coefficient vector of a polynomial is invariant under the action of $T_{\mathsf{SL}_k}$). Thus if a polynomial $w$ is stabilized by $T_{\mathsf{SL}_k}$, each monomial of $w$ is stabilized independently. A monomial is stabilized by $T_{\mathsf{SL}_k}$ iff each variable appears the same number of times. In degree $k$ there is only one monomial that has this property: $x_1 \cdots x_k$. □

**3.3 Proposition.** *For $\lambda \vdash kd$, $\mathrm{mult}_\lambda(\mathbb{C}[\mathsf{GL}_k x_1 \cdots x_k])_d = a_\lambda(k,d)$.*

*Proof.* We use the algebraic Peter-Weyl theorem. $\mathrm{mult}_\lambda(\mathbb{C}[\mathsf{GL}_k x_1 \cdots x_k])_d = \dim\{\lambda\}^{T_{\mathsf{SL}_k} \rtimes \mathfrak{S}_k} = \dim(\{\lambda\}^{T_{\mathsf{SL}_k}})^{\mathfrak{S}_k}$. Recall the vector space of tableaux with the Grassmann-Plücker relations. A basis of $\{\lambda\}$ is given by semistandard tableaux with entries $1, \ldots, k$. Each basis vector gets rescaled by the action of $T_{\mathsf{SL}_k}$. The $T_{\mathsf{SL}_k}$-invariants are the tableaux for which each number appears equally often. Since $\lambda \vdash kd$, each number appears exactly $d$ times. Taking the $\mathfrak{S}_k$-invariants of this space of tableaux, its dimension is precisely the plethysm coefficient $a_\lambda(k,d)$. □

CAVEAT: Let $b_\lambda(d,k) := \mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathsf{GL}_k(x_1 \cdots x_k)}]_d)$. We just saw $b_\lambda(d,k) \leq a_\lambda(k,d)$. It might be confusing that we also know $b_\lambda(d,k) \leq a_\lambda(d,k)$, because $\overline{\mathsf{GL}_k(x_1 \cdots x_k)}$ is a subvariety of $\mathrm{Sym}^k \mathbb{C}^k$.

## 3.2 Power sum

**3.4 Proposition.** *Let $m \geq 3$. Let $G = \mathsf{GL}_k$ and let $v = x_1^m + \cdots + x_k^m$. Then $\mathrm{stab}_G(v)$ is generated by the permutation matrices and the diagonal matrices with $m$th roots of unity on the main diagonal. Notation: $\mathbb{Z}_m^k \rtimes \mathfrak{S}_k$*

*Proof.* Clearly the listed matrices stabilize $v$. The rest of the proof uses partial derivatives. We postpone it for a few minutes. □

CAVEAT:

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}(x^2 + y^2) = \frac{1}{2}((x+y)^2 + (x-y)^2) = x^2 + y^2.$$

Let us therefore assume $m \geq 3$.

**3.5 Proposition.** *The power sum $x_1^m + \cdots + x_k^m$ is characterized by its stabilizer.*

*Proof.* The action of $\mathbb{Z}_m^k$ preserves the monomial structure. Thus if a polynomial $w$ is stabilized by $\mathbb{Z}_m^k$, each monomial is stabilized. A monomial is stabilized by $\mathbb{Z}_m^k$ iff each variable appears a multiple of $m$ times. In $\mathrm{Sym}^m \mathbb{C}^k$ there are only $k$ such monomials: $x_i^m$, $1 \leq i \leq k$. Invariance under $\mathfrak{S}_k$ ensures that they all have the same coefficient. Thus $w$ is a multiple of $x_1^m + \cdots + x_k^m$. ☐

The multiplicities $\mathbb{C}[\mathsf{GL}_k(x_1^m + \cdots + x_k^m)]$ can be determined, but it is a bit tricky and we postpone it for a few lectures to Section 3.6. But the simplest case goes as follows.

**3.6 Proposition.** *For $\lambda \vdash km$,* $\mathrm{mult}_\lambda\, \mathbb{C}[\mathsf{GL}_k(x_1^m + \cdots + x_k^m)] \geq a_\lambda(k, m)$.

*Proof.* We use the algebraic Peter-Weyl theorem. $\mathrm{mult}_\lambda(\mathbb{C}[\mathsf{GL}_k(x_1^m + \cdots + x_k^m)])_d = \dim\{\lambda\}^{\mathbb{Z}_m^k \rtimes \mathfrak{S}_k} = \dim(\{\lambda\}^{\mathbb{Z}_m^k})^{\mathfrak{S}_k}$. A basis of $\{\lambda\}$ is given by semistandard tableaux. Each basis vector gets rescaled by the action of $\mathbb{Z}_m^k$. The $\mathbb{Z}_m^k$-invariants are the tableaux for which each number appears a multiple of $m$ times often.

In particular we obtain the tableaux in which each number appears exactly $m$ times. Taking the $\mathfrak{S}_k$-invariants of this space of tableaux, its dimension is precisely the plethysm coefficient $a_\lambda(k, m)$. ☐

Remark: One can show that $\mathrm{mult}_\lambda\, \mathbb{C}[\overline{\mathsf{GL}_k(x_1^m + \cdots + x_k^m)}] = a_\lambda(k, m)$ for $\lambda \vdash km$.

*Rest of the proof of Proposition 3.4.* This is taken from [3].

Let $\vec{x} := (x_1, \ldots, x_k)$. Let $v := x_1^m + \cdots + x_k^m$. Define the *Hessian* $H_v(\vec{x})$ as the $k \times k$ matrix whose $(i, j)$-entry is

$$\frac{\partial^2}{\partial x_i \partial x_j} v(\vec{x}).$$

The matrix $H_v(\vec{x})$ is diagonal with entry $(i, i)$ being $m(m-1)x_i^{m-2}$. Thus

$$\det H_v(\vec{x}) = \prod_{i=1}^{k} m(m-1)x_i^{m-2}.$$

**3.7 Claim.** *$H_{g^{-1}v}(\vec{x}) = g^T \cdot H_v(g\vec{x}) \cdot g$. In particular $\det(H_{g^{-1}v}(\vec{x})) = \det(g)^2 \det(H_v(g\vec{x}))$.*

*Proof of claim.* Let $F := g^{-1}v$, i.e., $F(\vec{x}) = v(g\vec{x})$. We use the chain rule, $1 \leq i \leq k$:

$$\frac{\partial F}{\partial x_i}(\vec{x}) = \sum_{q=1}^{k} g_{q,i} \cdot \frac{\partial v}{\partial x_q}(g\vec{x}).$$

[3] Chen, Wigderson, Kayal, *Partial derivatives in arithmetic complexity and beyond*, Found. Trends Theor. Comput. Sci.. Chapter 2

We use the chain rule again:

$$\begin{aligned}\frac{\partial^2 F}{\partial x_i \cdot \partial x_j}(\vec{x}) &= \sum_{q=1}^{k} g_{q,i}\left(\sum_{p=1}^{k} g_{p,j}\cdot \frac{\partial^2 v}{\partial x_q \partial x_p}(g\vec{x})\right) \\ &= \sum_{1\le p,q\le k} g_{q,i}\cdot \frac{\partial^2 v}{\partial x_q \partial x_p}(g\vec{x})\cdot g_{p,j}.\end{aligned}$$

In matrix form: $H_F(\vec{x}) = g^T \cdot H_v(g\vec{x}) \cdot g$. □

Now let $v = g^{-1}v$. Then also the Hessians coincide:

$$H_v(\vec{x}) = H_{g^{-1}v}(\vec{x}) = g^T \cdot H_v(g\vec{x}) \cdot g.$$

In particular their determinants coincide:

$$\det(H_v(\vec{x})) = \det(g)^2 \det(H_v(g\vec{x})).$$

$$\prod_{i=1}^{k} m(m-1)x_i^{m-2} = \underbrace{\det(g)^2}_{\text{constant}} \prod_{i=1}^{k}\left(\sum_{j=1}^{k} g_{i,j}x_j\right)^{m-2}.$$

Now we use the uniqueness of factorization: Each $\sum_{j=1}^{k} g_{i,j}x_j$ is a scalar multiple of some $x_{i'}$. Thus $g$ has at most 1 nonzero entry in each column. Since $g \in \mathsf{GL}_k$, $g$ has at exactly 1 nonzero entry in each row and column. Clearly any permutation fixes $v$, so we can assume that $g$ is diagonal. The diagonal matrices that fix $v$ are precisely those whose diagonal entries are $m$th roots of unity. □

## 3.3 Determinant

We now discuss the determinant and the permanent. We prove that both are characterized by their stabilizer. One part of the proof requires some basic character theory, which we discuss in Section 3.5.

### The stabilizer

Let $X = (x_{i,j})$ be an $n \times n$ variable matrix.

$\det(gXh) = \det(g)\det(h)\det(X)$. Thus $\det(X) = \det(gXh)$ with $\det(g)\cdot\det(h) = 1$.

Moreover, $\det(X) = \det(gX^t h)$ with $\det(g)\cdot\det(h) = 1$.

These are the only symmetries, as was first shown by Frobenius in 1897[4]. Hence $\mathrm{stab}_{\mathsf{GL}_{n^2}}(\det_n) = (\mathsf{GL}_n \times \mathsf{GL}_n)/(\mathbb{C}^\times) \rtimes \mathbb{Z}_2$.

[4] Frobenius, *Über die Darstellung der endlichen Gruppen durch lineare Substitutionen.* Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin, pages 994–1015, 1897., §7, Satz I

### Characterization by the stabilizer

We want to see that $\det_n$ is characterized by its stabilizer. We need the following preliminary lemma.

**3.8 Lemma.** *From the exercises we know that every irreducible* $\mathsf{GL}_n$*-representation* $\{\lambda\}$ *is irreducible as an* $\mathsf{SL}_n$*-representation. We have that* $\{\lambda\}$ *is the trivial* $\mathsf{SL}_n$*-representation iff* $\lambda = n \times d$ *for some* $d$*.*

*Proof.* In order for $\{\lambda\}$ to be the trivial $\mathsf{SL}_n$-representation, we need $\{\lambda\}$ to be 1-dimensional. Recall that $\dim\{\lambda\}$ equals the number of semistandard tableaux of shape $\lambda$ with entries $1, \dots, n$. Thus the only 1-dimensional $\mathsf{GL}_n$-representations $\{\lambda\}$ satisfy $\lambda = n \times d$. Indeed, these correspond to the representation $gv = \det(g)^d.v$. In particular these are trivial with respect to the $\mathsf{SL}_n$-action. □

**3.9 Theorem.** $\det_n$ *is characterized by its stabilizer.*

*Proof.* $\otimes^n(\mathbb{C}^n \otimes \mathbb{C}^n)$ decomposes w.r.t. the $\mathsf{GL}_n \times \mathsf{GL}_n$-action (Schur-Weyl duality) as

$$\otimes^n(\mathbb{C}^n \otimes \mathbb{C}^n) = \bigoplus_{\lambda,\mu \vdash n} \{\lambda\} \otimes \{\mu\} \otimes [\lambda] \otimes [\mu].$$

Thus $\mathrm{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n)$ decomposes as

$$\mathrm{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n) = \bigoplus_{\lambda,\mu \vdash n} \{\lambda\} \otimes \{\mu\} \otimes ([\lambda] \otimes [\mu])^{\mathfrak{S}_n}.$$

Now we use $([\lambda] \otimes [\mu])^{\mathfrak{S}_n} = \begin{cases} \mathbb{C} & \text{iff } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$ (which we prove below using character theory):

$$\mathrm{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n) = \bigoplus_{\lambda \vdash n} \{\lambda\} \otimes \{\lambda\}.$$

Taking $\mathsf{SL}_n \times \mathsf{SL}_n$-invariants (Lemma 3.8):

$$(\mathrm{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n))^{\mathsf{SL}_n \times \mathsf{SL}_n} = \{1^n\} \otimes \{1^n\},$$

which is 1-dimensional. □

## 3.4 Permanent

### The stabilizer

Let $X = (x_{i,j})$ be an $n \times n$ variable matrix.

$\mathrm{per}(gXh) = \mathrm{per}(X)$ if $g$ and $h$ are permutation matrices.

Moreover, $\mathrm{per}(gXh) = \mathrm{per}(X)$ if $g$ and $h$ are diagonal matrices with $\det(gh) = 1$.

Moreover, $\mathrm{per}(X) = \mathrm{per}(X^t)$.

These are the only symmetries[5]. Hence $\mathrm{stab}_{\mathsf{GL}_{n^2}}(\mathrm{per}_n) = (Q_n \times Q_n)/\mathbb{C}^\times \rtimes \mathbb{Z}_2$, where $Q_n = T_n \rtimes \mathfrak{S}_n$.

### Characterization by the stabilizer

Analogously to $\det_n$ we see that $\mathrm{per}_n$ is characterized by its stabilizer.

**3.10 Theorem.** $\mathrm{per}_n$ *is characterized by its stabilizer.*

*Proof.* $\otimes^n(\mathbb{C}^n \otimes \mathbb{C}^n)$ decomposes w.r.t. the $\mathsf{GL}_n \times \mathsf{GL}_n$-action (Schur-Weyl duality) as

$$\otimes^n(\mathbb{C}^n \otimes \mathbb{C}^n) = \bigoplus_{\lambda,\mu \vdash n} \{\lambda\} \otimes \{\mu\} \otimes [\lambda] \otimes [\mu].$$

Thus $\mathrm{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n)$ decomposes as

$$\mathrm{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n) = \bigoplus_{\lambda,\mu \vdash n} \{\lambda\} \otimes \{\mu\} \otimes ([\lambda] \otimes [\mu])^{\mathfrak{S}_n}.$$

Now we use $([\lambda] \otimes [\mu])^{\mathfrak{S}_n} = \begin{cases} \mathbb{C} & \text{iff } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$ (which we prove below using character theory):

$$\mathrm{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n) = \bigoplus_{\lambda \vdash n} \{\lambda\} \otimes \{\lambda\}.$$

For $\lambda \vdash n$ Gay's theorem states that $\{\lambda\}^{T_n} = [\lambda]$, but this can easily be generalized: Let $T_{\mathsf{SL}_n} := T_n \cap \mathsf{SL}_n$. Then for $\lambda \vdash n$ we have $\{\lambda\}^{T_{\mathsf{SL}_n}} = [\lambda]$.

Thus taking $T_{\mathsf{SL}_n} \times T_{\mathsf{SL}_n}$-invariants we obtain:

$$(\mathrm{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n))^{T_{\mathsf{SL}_n} \times T_{\mathsf{SL}_n}} = \bigoplus_{\lambda \vdash n} [\lambda] \otimes [\lambda].$$

Taking $\mathfrak{S}_n \times \mathfrak{S}_n$-invariants yields

$$\bigoplus_{\lambda \vdash n} [\lambda]^{\mathfrak{S}_n} \otimes [\lambda]^{\mathfrak{S}_n} = [n] \otimes [n],$$

which is 1-dimensional. □

[5] Marcus and May, *The permanent function.* Canad. J. Math., 14:177–189, 1962.

## 3.5 Dual representations and character theory

In this section we discuss some basic character theory to prove the following statement, which is a missing part in our arguments in Sections 3.3 and 3.4.

**3.11 Proposition.** $\dim([\lambda]\otimes[\mu])^{\mathfrak{S}_n} = 1$ *iff* $\lambda = \mu$ *(0 otherwise).*

Remark: Using what we have learned last semester, with obstruction designs and explicit highest weight vectors it is easy to see that $\dim([\lambda]\otimes[\lambda])^{\mathfrak{S}_n} \geq 1$. For the corresponding hypergraph let the layer 1 hyperedges agree with the layer 2 hyperedges. This hypergraph decomposes into disjoint hypergraphs. The semigroup property implies that we only need to verify $([1^n]\otimes[1^n])^{\mathfrak{S}_n} > 0$. This can be verified directly by studying the $\mathfrak{S}_n$-action on the column tableau pair.

The complete proof of Prop. 3.11 can be easily seen using classical ideas from representation theory that we want to introduce now.

**3.12 Definition.** *Let $G$ be a group, $V$ be a finite dimensional vector space, and let $\varrho : G \to \mathsf{GL}(V)$ be a representation. Let $V^*$ be the dual space to $V$, i.e., the space of homogeneous linear forms on $V$. Then $V^*$ is a representation via $(gf)(x) := f(g^{-1}x)$, which is called the* dual representation *or the* contragredient representation*.*

**3.13 Lemma.** *Let $V$ be a $G$-representation. $V$ is irreducible iff $V^*$ is irreducible.*

*Proof.* Let $V^*$ be irreducible. Let $W \subseteq V$ by a $G$-subrepresentation, in particular a linear subspace. Then the vanishing ideal $I(W)_1$ in degree 1 is called the *annihilator* $W^\perp$, which is a $G$-subrepresentation of $V^*$:

$$W^\perp := \{f \in V^* : f(W) = \{0\}\},$$

Since $V^*$ is irreducible, either $W^\perp = 0$ or $W^\perp = V^*$. If $W^\perp = V^*$, then all linear polynomials vanish on $W$. Since $W$ is a linear subspace, $W = 0$. If $W^\perp = 0$, then no linear polynomial vanishes on $W$. Since $W$ is a linear subspace, $V = W$. In both cases $W$ is a trivial $G$-subrepresentation of $V$. Thus $V$ is irreducible.

We finish the argument by showing that $V^{**} = V$ are isomorphic $G$-representations.

$$V^{**} = \{\Phi : V^* \to \mathbb{C} \mid \Phi \text{ linear}\}$$

The canonical isomorphism $\Xi : V \to V^{**}$ is known from linear algebra as follows:

$$(\Xi(v))(\varphi) := \varphi(v), \quad v \in V, \varphi \in V^*. \tag{†}$$

But $\Xi$ is $G$-equivariant:

$$(\Xi(gv))(\varphi) \overset{(\dagger)}{=} \varphi(gv) = (g^{-1}\varphi)(g) \overset{(\dagger)}{=} (\Xi(v))(g^{-1}\varphi) = (g(\Xi(v)))(\varphi),$$

i.e., $\Xi(gv) = g(\Xi(v))$. □

If $V$ is irreducible of type $\lambda$, then we denote by $\lambda^*$ the type of $V^*$.

The following two propositions prove Proposition 3.11.

**3.14 Proposition** (A)**.** *Let $G$ be a linearly reductive group and $W$ be a $G$-representation and let $\{\lambda\}$ denote the irreducible $G$-representation of type $\lambda$. Then*

$$\mathrm{mult}_\lambda(W) = \dim((\{\lambda^*\} \otimes W)^G).$$

*In particular* $\dim((\{\lambda\}^* \otimes \{\lambda\})^G) = 1$ *and for* $\lambda \neq \mu$ *we have* $\dim((\{\mu\}^* \otimes \{\lambda\})^G) = 0$.

We prove this in Section (A).

**3.15 Proposition** (B)**.** *$[\lambda]$ and $[\lambda]^*$ are isomorphic Specht modules.*

We prove this in Section (B).

CAVEAT: Proposition 3.15 holds for the symmetric group, but it is false for example for $\mathsf{GL}_n$, $T_n$, or the cyclic group of order $> 2$.

*Proof of Proposition 3.11.*

$$\dim([\lambda] \otimes [\mu])^{\mathfrak{S}_n} \overset{(B)}{=} \dim([\lambda^*] \otimes [\mu])^{\mathfrak{S}_n} \overset{(A)}{=} \mathrm{mult}_{[\lambda]}([\mu]) = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$ □

### (A) "Contraction" of representations: Proof of Proposition 3.14

**3.16 Lemma.** *Given two $G$-representations $V$ and $W$,* $\mathrm{Hom}(V, W) = V^* \otimes W$ *is a $G \times G$-representation via*

$$((g', g)\varphi)(v) := g'(\varphi(g^{-1}v)).$$

*Proof.* We prove $((g'h', gh))\varphi = (g', g)((h', h)\varphi)$.

$$((g'h', gh)\varphi)(v) \quad = \quad (g'h')\varphi((gh)^{-1}v) = g'(h'\varphi(h^{-1}(g^{-1}v))) = g(\underbrace{(h', h)\varphi}_{=:\Psi})(g^{-1}v) = ((g', g)\Psi)(v)$$

□

**3.17 Lemma.** *Embed $G \hookrightarrow G \times G$, $g \mapsto (g, g)$. In this way* $\mathrm{Hom}(V, W)$ *is a $G$-representation. Its invariant space is the space of equivariant linear maps:* $\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W)$.

*Proof.* Recall that $\varphi \in \mathrm{Hom}_G(V, W)$ iff $g(\varphi(v)) = \varphi(gv)$ for all $g \in G$.

If $\varphi \in \mathrm{Hom}(V, W)$ is $G$-invariant, then $\varphi(v) = g(\varphi(g^{-1}v))$ and thus

$$g^{-1}(\varphi(v)) = (g^{-1}g)\varphi(g^{-1}v) = \varphi(g^{-1}v),$$

hence $\varphi$ is $G$-equivariant. The proof works analogously in the other direction. □

Proposition 3.14 is now readily proved:

Schur's lemma says that

$$\dim(\mathrm{Hom}_G(\{\lambda\}, W)) = \mathrm{mult}_\lambda(W).$$

We just saw that

$$\mathrm{Hom}_G(\{\lambda\}, W) = (\mathrm{Hom}(\{\lambda\}, W))^G = (\{\lambda\}^* \otimes W)^G.$$

Therefore

$$\mathrm{mult}_\lambda(W) = \dim(\{\lambda\}^* \otimes W)^G.$$

□

**(B) Some character theory of finite groups: The Specht modules are self-dual (Prop. 3.15)**

We will use character theory to prove that the Specht modules $[\lambda]$ and $[\lambda]^*$ are isomorphic. Character theory is also currently the most efficient tool to compute Kronecker coefficients.

**3.18 Definition.** *Let $\varrho : G \to \mathsf{GL}(V)$ be a representation. Then the map*

$$\chi_\varrho : G \to \mathbb{C}, \quad \chi_\varrho(g) = \mathrm{tr}(\varrho(g))$$

*is called the* character *of $\varrho$.*

**3.19 Observation.** *The character is a function that is constant on conjugacy classes. We call these functions* class functions*.*

*Proof.* $\chi_\varrho(h^{-1}gh) = \mathrm{tr}(\varrho(h^{-1}gh)) = \mathrm{tr}(\varrho(h^{-1})\varrho(g)\varrho(h))) = \mathrm{tr}(\varrho(h)\varrho(h^{-1})\varrho(g))) = \mathrm{tr}(\varrho(hh^{-1})\varrho(g))) = \mathrm{tr}(\varrho(g)) = \chi_\varrho(g)$. □

**3.20 Proposition.** *Isomorphic representations have coinciding characters. (We will prove the other direction later)*

*Proof.* Let $(V, \varrho_V)$ and $(W, \varrho_W)$ be isomorphic representations with isomorphism $\gamma : V \to W$. As a product of matrices this means $\gamma\varrho_V(g)v = \varrho_W(g)\gamma v$ for all $v \in V$, thus $\gamma\varrho_V(g) = \varrho_W(g)\gamma$. In other words

$$\gamma\varrho_V(g)\gamma^{-1} = \varrho_W(g).$$

Thus the traces of $\varrho_V(g)$ and $\varrho_W(g)$ coincide. □

**3.21 Example.** *We calculate the characters of $\mathfrak{S}_3$.*

| | $id$ | $(12)$ | $(123)$ |
|---|---|---|---|
| $[(3)]$ | 1 | 1 | 1 |
| $[(1^3)]$ | 1 | $-1$ | 1 |
| $[(2,1)]$ | ? | ? | ? |

*Let* $a := \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, $b := \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$.

$$\mathrm{id}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix},$$

*thus*

$$\varrho_{(2,1)}(\mathrm{id}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

*Therefore* $\chi_{(2,1)}(\mathrm{id}) = \mathrm{tr}(\varrho_{(2,1)}(\mathrm{id})) = 2$.

$$(12)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a-b \\ -b \end{pmatrix},$$

*thus*

$$\varrho_{(2,1)}((12)) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

*Therefore* $\chi_{(2,1)}((12)) = 0$.

$$(123)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a-b \end{pmatrix},$$

*thus*

$$\varrho_{(2,1)}((12)) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

*Therefore* $\chi_{(2,1)}((123)) = -1$.

| | *id* | (12) | (123) |
|---|---|---|---|
| $[(3)]$ | 1 | 1 | 1 |
| $[(1^3)]$ | 1 | $-1$ | 1 |
| $[(2,1)]$ | 2 | 0 | $-1$ |

*This is called a* character table.

From this example we see that in the case of the symmetric group the characters are integers.

**3.22 Lemma.** $\varrho^*(g) = \varrho(g)^{-t}$ *(i.e., the representation matrix of the dual is the transpose inverse matrix).*

*Proof.* Instead of $\varrho(g) : V \to V$ we start out more generally and let $D : V \to W$ be a linear map, not necessarily an isomorphism. The transpose $D^t : W^* \to V^*$ is defined via

$$f \mapsto f \circ D$$

or, in other words

$$(D^t(f))(x) = f(D(x)) \text{ for all } x \in V.$$

If $D$ is invertible, we can analyze $D^{-1}$ instead of $D$:

$$(D^{-t}(f))(x) = f(D^{-1}(x)).$$

If $D = \varrho(g)$, then the right-hand side becomes $f(g^{-1}x)$, which means that the left-hand side is equal to $(gf)(x)$. □

**3.23 Theorem.** *Let $g \in G$ have finite order (e.g., if $G$ is finite). The character of the dual representation is the complex conjugate: $\chi^*(g) = \overline{\chi(g)}$.*

*Proof.* Since $\varrho(g)$ is of finite order, $\varrho(g)$ is diagonalizable. Let $d = A^{-1}\varrho(g)A$ be diagonal. Then $d^{-1} = A^{-1}\varrho(g)^{-1}A$ (invert all three and switch the order). Clearly $\chi(g) := \operatorname{tr}(\varrho(g)) = \operatorname{tr}(d)$ and $\chi^*(g) \overset{\text{Lemma 3.22}}{=} \operatorname{tr}(\varrho(g)^{-t}) = \operatorname{tr}(\varrho(g)^{-1}) = \operatorname{tr}(d^{-1})$. Since $d$ has finite order, the entries on the diagonal of $d$ are roots of unity. Thus $d^{-1} = \overline{d}$. Summing up the trace gives: $\chi^*(g) = \overline{\chi(g)}$. □

Remark: We know from our explicit description of the Specht modules (straightening algorithm) that the entries of the representation matrices $\varrho(g)$, $g \in \mathfrak{S}_n$, are all real-valued. Thus the characters of the symmetric group are real-valued. Thus $[\lambda]^* = [\lambda]$.

**3.24 Corollary.** *If $g$ has finite order, then $\chi(g^{-1}) = \overline{\chi(g)}$.*

*Proof.*

$$\chi(g^{-1}) = \operatorname{tr}(\varrho(g^{-1})) = \operatorname{tr}(\varrho(g)^{-1}) = \operatorname{tr}(\varrho(g)^{-t}) \overset{\text{Lem. 3.22}}{=} \operatorname{tr}\varrho^*(g) = \chi^*(g) \overset{\text{Thm. 3.23}}{=} \overline{\chi(g)}.$$

□

### Characters characterize the representation

Let $G$ be finite.

For a linear map $\varphi : U \to V$ define a $G$-morphism $\tilde{\varphi} : U \to V$ via

$$\tilde{\varphi}(u) = \frac{1}{|G|}\sum_{g\in G} g\varphi(g^{-1}u) \qquad = \qquad \frac{1}{|G|}\sum_{g\in G} \varrho_V(g)\varphi(\varrho_U(g^{-1})u).$$

In matrix presentation:

$$\tilde{\varphi} = \frac{1}{|G|}\sum_{g\in G} \varrho_V(g)\cdot\varphi\cdot\varrho_U(g^{-1}).$$

**3.25 Corollary.** *Let $U, V$ be irreducible $G$-representations and $\varphi : U \to V$ a linear map.*

1. *If $U \not\cong V$, then $\tilde{\varphi} = 0$.*
2. *If $U = V$, then $\tilde{\varphi} = \frac{\operatorname{tr}\varphi}{n}\operatorname{id}_U$.*

*Proof.* The first claim is clear by Schur's lemma, because $\tilde{\varphi}$ is a $G$-morphism. Furthermore, also by Schur's lemma, if $U = V$, then $\tilde{\varphi} = \alpha\operatorname{id}_U$.

$$n\alpha = \operatorname{tr}(\alpha\operatorname{id}_U) = \operatorname{tr}(\tilde{\varphi}) = \frac{1}{|G|}\sum_{g\in G}\underbrace{\operatorname{tr}(\varrho_U(g)\cdot\varphi\cdot\varrho_U(g^{-1}))}_{=\operatorname{tr}\varphi} = \operatorname{tr}(\varphi).$$

Thus $\alpha = \frac{\operatorname{tr}(\varphi)}{n}$. □

**3.26 Corollary.** *Let $U, V$ be irreducible $G$-representations and $R(g) := \varrho_U(g)$, $S(g) := \varrho_V(g)$ are the representation matrices.*

1. *If $U \not\simeq V$, then $\forall i,j,k,l : \frac{1}{|G|}\sum_{g\in G} S(g)_{ij}R(g^{-1})_{kl} = 0$*
2. *If $U = V$, $\dim U = n$, then $\forall i,j,k,l : \frac{1}{|G|}\sum_{g\in G} R(g)_{ij}R(g^{-1})_{kl} = \frac{1}{n}\delta_{il}\delta_{jk}$.*

*Proof.* Let $E_{jk}$ be the zero matrix with a single 1 in row $j$, column $k$. For matrices $S, R$ we have

$$(S\cdot E_{jk}\cdot R)_{il} = \sum_{a,b} S_{ia}(E_{jk})_{ab}R_{bl} = S_{ij}R_{kl}. \qquad (\dagger)$$

We want to apply Cor. 3.25 to $\varphi = E_{jk}$:

$$\tilde{\varphi}_{il} = \left(\frac{1}{|G|}\sum_{g\in G} S(g)E_{jk}R(g^{-1})\right)_{il} = \frac{1}{|G|}\sum_{g\in G}\big(S(g)E_{jk}R(g^{-1})\big)_{il} \overset{(\dagger)}{=} \frac{1}{|G|}\sum_{g\in G} S(g)_{ij}R(g^{-1})_{kl}.$$

If $U \not\simeq V$, then Cor. 3.25 implies $\tilde{\varphi}_{il} = 0$, which proves the first claim. If $U = V$, then Cor. 3.25 implies $\tilde{\varphi}_{il} = \frac{\operatorname{tr} E_{jk}}{n}\delta_{il} = \frac{1}{n}\delta_{jk}\delta_{il}$, which proves the second claim. □

**3.27 Definition.** *Let $\varphi, \psi$ be functions $G \to \mathbb{C}$. We define*

$$\langle\varphi,\psi\rangle := \frac{1}{|G|}\sum_{g\in G}\varphi(g)\overline{\psi(g)}.$$

**3.28 Remark.** *$\langle .,.\rangle$ is an inner product on the complex vector space of functions $G\to\mathbb{C}$. An* orthonormal system *is a set $\{\chi_1,\dots,\chi_k : G\to\mathbb{C}\}$ such that $\langle\chi_i,\chi_j\rangle = \delta_{ij}$. Every orthonormal system is linearly independent.*

**3.29 Theorem** (Fundamental theorem (orthogonality relations))**.** *Let $U$ and $V$ be irreducible $G$-representations. Then*

$$\langle\chi_U,\chi_V\rangle = \begin{cases} 1 & \text{if } U\simeq V\\ 0 & \text{otherwise}\end{cases}$$

*Proof.*

$$\begin{aligned}\langle\chi_U,\chi_V\rangle &= \frac{1}{|G|}\sum_{g\in G}\chi_U(g)\overline{\chi_V(g)} \overset{\text{Cor. 3.24}}{=} \frac{1}{|G|}\sum_{g\in G}\chi_U(g)\chi_V(g^{-1})\\ &= \frac{1}{|G|}\sum_{g\in G}(\sum_i S_{ii}(g))(\sum_j R_{jj}(g^{-1})) = \sum_{i,j}\underbrace{\frac{1}{|G|}\sum_{g\in G}S_{ii}(g)R_{jj}(g^{-1})}_{\text{cp. Cor. 3.26}}.\end{aligned}$$

If $U \not\simeq V$, this is 0 by Corollary 3.26.
If $U = V$, then Corollary 3.26 gives

$$\langle\chi_U,\chi_V\rangle = \sum_{i,j}\frac{1}{n}\delta_{ij}\delta_{ij} = \sum_{i=1}^{n}\frac{1}{n} = 1.$$ □

**3.30 Theorem.** *G-representations are isomorphic iff their characters coincide.*

*Proof.* One direction is known from Prop. 3.20. For the other direction, let $U = \bigoplus_\lambda c_\lambda\{\lambda\}$ be a decomposition into irreducible representations, and analogously for $V = \bigoplus_\lambda d_\lambda\{\lambda\}$. Then $\chi_U = \sum_\lambda c_\lambda \chi_\lambda$ and $\chi_V = \sum_\lambda d_\lambda \chi_\lambda$ for some natural numbers $c_\lambda, d_\lambda$. Since $\{\chi_\lambda\}$ is an orthonormal system, it is linearly independent. Therefore $c_\lambda = d_\lambda$ for all $\lambda$. We conclude $U \simeq V$. □

*Proof of Prop. 3.15.* Let $V = [\lambda]$ with character $\chi_V$. Since $V$ is a Specht module, $\chi_V(g) \in \mathbb{R}$. Thus $\chi_{V^*}(g) \overset{\text{Thm. 3.23}}{=} \overline{\chi_V(g)} = \chi_V(g)$. Hence $\chi_V = \chi_{V^*}$. Using Theorem 3.30 we conclude that $V \simeq V^*$. □

## 3.6 Power sum revisited

Since we got a bit more familiar with invariant spaces, we now determine the multiplicities in the coordinate rings of the power sum and the unit tensor. We start with the simpler case of the power sum.

We use the notation in the STOC 2011 paper by Bürgisser and Ikenmeyer, so $m$ is now the number of variables and $D$ is the degree. The formulas in this and the next section are unpublished calculations by Ikenmeyer and Panova. Parts also appear in a preprint of Nishiyama.

The power sum is the polynomial $x_1^D + \cdots + x_m^D$. Let $H := \mathbb{Z}_D^m \rtimes \mathfrak{S}_m$ denote its stabilizer. Let $\lambda \vdash Dd$.

If $\varrho \vdash_m d$ is a partition, then the frequency notation $\kappa \in \mathbb{N}^m$ is defined via

$$\kappa_i = |\{j \mid \varrho_j = i\}|.$$

E.g., the frequency notation of $\varrho = (3,3,2,0)$ is $(0,1,2,0)$. We observe that $|\varrho| = \sum_i i\kappa_i$.

We group $\mathfrak{S}_m$ acts on $\mathbb{N}^m$ by permuting the positions. Note that under this action we have $\operatorname{stab}\varrho = \mathfrak{S}_{\kappa_1} \times \mathfrak{S}_{\kappa_2} \times \cdots \times \mathfrak{S}_{\kappa_m}$.

**3.31 Theorem.** $\dim\{\lambda\}^H = \sum_{\varrho \vdash_m d} \sum_{\substack{\mu^1,\mu^2,\ldots,\mu^d \\ \mu^i \vdash \kappa_i D i}} c^\lambda_{\mu^1,\mu^2,\ldots,\mu^d} \prod_{i=1}^d a_{\mu^i}(\kappa_i, iD)$, *where $\kappa$ is the frequency notation of $\varrho$, and $c^\lambda_{\mu^1,\mu^2,\ldots,\mu^d}$ is the* multi-Littlewood-Richardson coefficient *that denotes the multiplicity of $\{\lambda\}$ in the tensor product $\{\mu^1\} \otimes \ldots \otimes \{\mu^d\}$.*

*Proof.*

$$\{\lambda\}^H = (\{\lambda\}^{\mathbb{Z}_D^m})^{\mathfrak{S}_m} = \left( \bigoplus_{\substack{\gamma \in \mathbb{N}^m \\ |\gamma| = d}} [\lambda]^{G_\gamma} \right)^{\mathfrak{S}_m}$$

where for $\gamma \in \mathbb{N}^m, |\gamma| = d$, $G_\gamma \subseteq \mathfrak{S}_{dD}$ is defined as the Young subgroup $\mathfrak{S}_{\gamma_1 D} \times \cdots \times \mathfrak{S}_{\gamma_m D}$. The last equality can be seen using the tableau bases on both sides.

For a partition $\varrho \vdash_m d$ let $\mathfrak{S}_m \varrho \subseteq \mathbb{N}^m$ denote the orbit of $\varrho$. Note that $\varrho$ is the only partition in its orbit, while the other lists are not in the correct order. Grouping the RHS in the previous equation we obtain

$$\bigoplus_{\varrho \vdash_m d} \left( \bigoplus_{\gamma \in \mathfrak{S}_m \varrho} [\lambda]^{G_\gamma} \right)^{\mathfrak{S}_m},$$

so we can study each $\varrho$ independently.

Let $\operatorname{stab} \varrho \leq \mathfrak{S}_m$ denote the stabilizer of $\varrho$.

**3.32 Claim.** $\dim \left( \bigoplus_{\gamma \in \mathfrak{S}_m \varrho} [\lambda]^{G_\gamma} \right)^{\mathfrak{S}_m} = \dim \left( [\lambda]^{G_\varrho} \right)^{\operatorname{stab} \varrho}$.

*Proof.* We construct an isomorphism of vector spaces.

Let $W^\varrho := [\lambda]^{G_\varrho}$ and $W_\varrho := \bigoplus_{\gamma \in \mathfrak{S}_m \varrho} W^\gamma$. Let $\pi_1, \dots, \pi_r$ be a system of representatives of left cosets for $\operatorname{stab} \varrho \leq \mathfrak{S}_m$ with $\pi_1 = \mathrm{id}$, i.e., $\mathfrak{S}_m = \pi_1 \operatorname{stab} \varrho \,\dot\cup\, \cdots \,\dot\cup\, \pi_r \operatorname{stab} \varrho$ and we have $\mathfrak{S}_m \varrho = \{\pi_1 \varrho, \dots, \pi_r \varrho\}$. Therefore we have the decomposition

$$W_\varrho = \bigoplus_{j=1}^{r} \pi_j W^\varrho.$$

Let $\overline{p} : W_\varrho \twoheadrightarrow W^\varrho$ be the projection according to this decomposition. We claim that the restriction

$$p : (W_\varrho)^{\mathfrak{S}_m} \to (W^\varrho)^{\operatorname{stab} \varrho}$$

is an isomorphism of vector spaces. This then finishes the proof. We verify well-definedness, injectivity, and surjectivity of $p$.

Well-definedness: The spaces $\pi_1 W^\varrho, \dots, \pi_r W^\varrho$ are permuted by $\mathfrak{S}_m$. Every $\sigma \in \operatorname{stab} \varrho$ fixes $W^\varrho$, thus $\sigma v_1 = v_1$ if $v_1 \in W^\varrho$. Thus the map $v = \sum_{j=1}^r v_j \overset{\overline{p}}{\mapsto} v_1$ maps $W_\varrho$ to $(W^\varrho)^{\operatorname{stab} \varrho}$.

Injectivity: If $v \in (W_\varrho)^{\mathfrak{S}_m}$, then $v = \pi v = \sum_j \pi v_j$. Therefore $v_j = \pi_j v_1$. If $p(v) = 0$, then $v_1 = 0$, thus all $v_j = 0$, which proves injectivity.

Surjectivity: Let $v_1 \in (W^\varrho)^{\operatorname{stab} \varrho}$. Set $v_j := \pi_j v_1$ and put $v := \sum_j v_j$. Clearly $p(v) = v_1$. It remains to verify that $v$ is $\mathfrak{S}_m$-invariant.

$$v = \sum_{j=1}^{r} \pi_j v_1 = \sum_{j=1}^{r} \tfrac{1}{|\operatorname{stab} \varrho|} \sum_{\tau \in \operatorname{stab} \varrho} \pi_j \tau v_1 = \tfrac{1}{|\operatorname{stab} \varrho|} \sum_{\pi \in \mathfrak{S}_m} \pi v_1,$$

which is $\mathfrak{S}_m$-invariant. $\square$

We are left with determining $\dim \left( [\lambda]^{G_\varrho} \right)^{\operatorname{stab} \varrho}$.

$$\dim \left( [\lambda]^{G_\varrho} \right)^{\operatorname{stab} \varrho} = \dim \mathrm{HWV}_\lambda(\{\lambda\} \otimes ([\lambda]^{G_\varrho})^{\operatorname{stab} \varrho}) = \dim \mathrm{HWV}_\lambda((\otimes^{dD} V)^{G_\varrho \rtimes \operatorname{stab} \varrho})$$

$$
\begin{aligned}
(\otimes^{dD}V)^{G_\varrho \rtimes \operatorname{stab}\varrho} &= (\operatorname{Sym}^{D\varrho_1} V \otimes \cdots \otimes \operatorname{Sym}^{D\varrho_m} V)^{\operatorname{stab}\varrho} \\
&= (\bigotimes^{\kappa_1} \operatorname{Sym}^{D} V \otimes \bigotimes^{\kappa_2} \operatorname{Sym}^{2D} V \otimes \cdots \otimes \bigotimes^{\kappa_d} \operatorname{Sym}^{dD} V)^{\operatorname{stab}\varrho} \\
&= \underbrace{\operatorname{Sym}^{\kappa_1} \operatorname{Sym}^{D} V}_{=\bigoplus_{\mu^1}\{\mu^1\}^{\oplus a_{\mu^1}(\kappa_1, D)}} \otimes \operatorname{Sym}^{\kappa_2} \operatorname{Sym}^{2D} V \otimes \cdots \otimes \underbrace{\operatorname{Sym}^{\kappa_d} \operatorname{Sym}^{dD} V}_{=\bigoplus_{\mu^d}\{\mu^d\}^{\oplus a_{\mu^d}(\kappa_d, dD)}} \qquad (\dagger)
\end{aligned}
$$

where $\kappa$ is the frequency notation of $\varrho$. The multiplicity of $\{\mu^i\}$ in $\operatorname{Sym}^{\kappa_i} \operatorname{Sym}^{iD} V$ is $a_{\mu^i}(\kappa_i, iD)$. Let $c^{\lambda}_{\mu^1,\mu^2,\ldots,\mu^d}$ denote the multiplicity of $\{\lambda\}$ in the tensor product $\{\mu^1\} \otimes \ldots \otimes \{\mu^d\}$. Using distributivity we obtain that the multiplicity of $\{\lambda\}$ in the representation $(\dagger)$ equals

$$
\sum_{\substack{\mu^1,\mu^2,\ldots,\mu^d \\ \mu^i \vdash \kappa_i D i}} c^{\lambda}_{\mu^1,\mu^2,\ldots,\mu^d} \prod_{i=1}^{d} a_{\mu^i}(\kappa_i, iD)
$$

We conclude

$$
\dim\{\lambda\}^H = \sum_{\varrho \vdash_m d} \sum_{\substack{\mu^1,\mu^2,\ldots,\mu^d \\ \mu^i \vdash \kappa_i D i}} c^{\lambda}_{\mu^1,\mu^2,\ldots,\mu^d} \prod_{i=1}^{d} a_{\mu^i}(\kappa_i, iD).
$$

□

## 3.7 Unit tensor

The unit tensor $\sum_{i=1}^m e_i \otimes e_i \otimes e_i$ has properties similar to the power sum. Its stabilizer in $\mathsf{GL}_m^3$ is $H := D_m \rtimes \mathfrak{S}_m$, where

$$
D_m := \{(\operatorname{diag}(\alpha_1^{(1)}, \ldots, \alpha_m^{(1)}), \ldots, \operatorname{diag}(\alpha_1^{(3)}, \ldots, \alpha_m^{(3)})) \mid \forall i : \alpha_i^{(1)} \alpha_i^{(2)} \alpha_i^{(3)} = 1\}.
$$

On the homework sheet we saw that the unit tensor is characterized by its stabilizer. Using the algebraic Peter-Weyl theorem we determine the multiplicities in the coordinate ring of the orbit of the unit tensor.

**3.33 Theorem.** $\dim\{\lambda,\lambda',\lambda''\}^H = \sum_{\varrho \vdash_m d} \sum_{\beta,\beta',\beta''} j_{\beta,\varrho}(\lambda) j_{\beta',\varrho}(\lambda') j_{\beta'',\varrho}(\lambda'') \left(\prod_{i=1}^m k(\beta^i, \beta'^i, \beta''^i)\right)$, *where for $\kappa$ being the frequency notation of $\varrho$*

- *the sum for $\beta$ is over all lists of partitions such that $\beta^i \vdash \kappa_i$ and analogously for $\beta'$ and $\beta''$, and*
- $j_{\beta,\varrho}(\lambda) := \sum_{\substack{\nu^1,\ldots,\nu^m \\ \nu^i \vdash i\kappa_i}} c^{\lambda}_{\nu^1,\ldots,\nu^m} \left(\prod_{i=1}^m a_{\nu^i}(\beta^i, i)\right)$,

*Proof.* $\{\lambda,\lambda',\lambda''\} = \{\lambda\} \otimes \{\lambda'\} \otimes \{\lambda''\}$.

$\{\lambda,\lambda',\lambda''\}$ has a basis given by triples of tableaux and $D_m$ rescales basis vectors. Thus a vector is invariant if all basis vectors in its support are invariant.

$D_m$ contains the subgroup

$$\{(\mathrm{diag}(\alpha, 1, 1, \dots, 1), \mathrm{diag}(\alpha^{-1}, 1, 1, \dots, 1), \mathrm{id})\}$$

and all other such subgroups where $\alpha$ and $\alpha^{-1}$ are both on position $i$ on two different diagonals. A basis vector is invariant under these groups if all three tableaux have the same content. Since $D_m$ is generated by these groups, this precisely characterizes the invariants: $\{\lambda, \lambda', \lambda''\}^{D_m}$ has as a basis those triples of tableaux in which all three tableaux share the same content $\gamma \in \mathbb{N}^m$, $|\gamma| = d$:

$$\{\lambda, \lambda', \lambda''\}^{D_m} = \bigoplus_{\substack{\gamma \in \mathbb{N}^m,\\ |\gamma|=d}} \{\lambda\}^\gamma \otimes \{\lambda'\}^\gamma \otimes \{\lambda''\}^\gamma,$$

where $\{\lambda\}^\tau$ denotes the vector space of tableaux of shape $\lambda$ and content $\tau$.

$\bigoplus_{\gamma \in \mathfrak{S}_m \tau} \{\lambda\}^\gamma$ is an $\mathfrak{S}_m$-representation. As seen in the proof for the power sum, we group together with respect to the content:

$$(\{\lambda, \lambda', \lambda''\}^{D_m})^{\mathfrak{S}_m} = \bigoplus_{\varrho \vdash_m d} ( \bigoplus_{\gamma \in \mathfrak{S}_m \varrho} \{\lambda\}^\gamma \otimes \{\lambda'\}^\gamma \otimes \{\lambda''\}^\gamma)^{\mathfrak{S}_m}$$

Completely analogously to the proof for the power sum, we can take $\mathrm{stab}\,\varrho$-invariants instead of $\mathfrak{S}_m$-invariants:

$$\dim( \bigoplus_{\gamma \in \mathfrak{S}_m \varrho} \{\lambda\}^\gamma \otimes \{\lambda'\}^\gamma \otimes \{\lambda''\}^\gamma)^{\mathfrak{S}_m} = \dim(\{\lambda\}^\varrho \otimes \{\lambda'\}^\varrho \otimes \{\lambda''\}^\varrho)^{\mathrm{stab}\,\varrho}$$

We analyze the action of $\mathrm{stab}\,\varrho$ separately on each of the three tableau spaces, i.e., we decompose $\{\lambda\}$, $\{\lambda'\}$, and $\{\lambda''\}$ as $\mathrm{stab}\,\varrho$-representations. Once this is done, Kronecker coefficients determine the $\mathrm{stab}\,\varrho$-invariant space dimension.

As seen in the proof for the power sum:

**3.34 Claim.**

$$\{\lambda\}^\varrho \overset{\mathrm{stab}\,\varrho\text{-}repr}{\simeq} \bigoplus_{\substack{\beta^1,\dots,\beta^m\\ \beta^i \vdash \kappa_i}} \underbrace{\sum_{\substack{\nu^1,\dots,\nu^m\\ \nu^i \vdash i\kappa_i}} c^\lambda_{\nu^1,\dots,\nu^m} \left( \prod_{i=1}^m a_{\nu^i}(\beta^i, i) \right)}_{=: j_{\beta,\varrho}(\lambda)} [\beta^1] \otimes \dots \otimes [\beta^m],$$

*where $\kappa$ is the frequency notation of $\varrho$.*

*Proof.* Note that $\{\lambda\}^{m \times k} = \bigoplus_{\mu \vdash m} a_\lambda(\mu, k)[\mu]$, as a generalization of Gay's theorem (this can be taken as the definition of the generalized plethysm coefficient).

We first prove $(*)$: $\bigotimes^i \mathrm{Sym}^j V = (\bigotimes^{ij} V)^{\mathfrak{S}^i_j} = \bigoplus_{\nu \vdash ij} \{\nu\} \otimes [\nu]^{\mathfrak{S}^i_j} = \bigoplus_{\nu \vdash ij} \{\nu\} \otimes \{\nu\}^{i \times j} = \bigoplus_{\nu \vdash ij, \varphi \vdash j} a_\nu(\varphi, j) \{\nu\} \otimes [\varphi]$, where for the last equality we use the generalized Gay's theorem.

Now we can calculate:

$$
\begin{aligned}
\bigoplus_{\lambda\vdash d}\{\lambda\}\otimes\{\lambda\}^{\varrho} &= \bigoplus_{\lambda\vdash d}\{\lambda\}\otimes[\lambda]^{G_\varrho} = (\otimes^d V)^{G_\varrho} = \mathrm{Sym}^{\varrho_1} V\otimes\cdots\otimes \mathrm{Sym}^{\varrho_m} V\\
&= \bigotimes^{\kappa_1}\mathrm{Sym}^1 V \quad \otimes \quad \cdots \quad \otimes \quad \bigotimes^{\kappa_d}\mathrm{Sym}^d V\\
&\overset{(*)}{=} \bigoplus_{\substack{\nu^1\vdash 1\kappa_1\\ \beta\vdash\kappa_1}} a_{\nu^1}(\beta^1,1)\{\nu^1\}\otimes[\beta^1] \quad\otimes\quad\cdots\quad\otimes\quad \bigoplus_{\substack{\nu^d\vdash d\kappa_d\\ \beta\vdash\kappa_d}} a_{\nu^d}(\beta^d,d)\{\nu^d\}\otimes[\beta^d]\\
&= \bigoplus_{\nu,\beta}\Big(\prod_{i=1}^m a_{\nu^i}(\beta^i,i)\Big)(\{\nu^1\}\otimes\{\nu^m\})\otimes[\beta^1]\otimes\cdots\otimes[\beta^m].
\end{aligned}
$$

Taking HWVs of weight $\lambda$ on both sides we obtain

$$
\{\lambda\}^{\varrho} = \bigoplus_{\nu,\beta} c^{\lambda}_{\nu^1,\dots,\nu^m}\Big(\prod_{i=1}^m a_{\nu^i}(\beta^i,i)\Big)[\beta^1]\otimes\cdots\otimes[\beta^m].
$$

□

Since the dimension of the $\mathfrak{S}_{\kappa_i}$-invariant space of $[\beta^i]\otimes[\beta'^i]\otimes[\beta''^i]$ is given by the Kronecker coefficient $k(\beta^i,\beta'^i,\beta''^i)$, we obtain:

$$
\dim(\{\lambda\}^{\varrho}\otimes\{\lambda'\}^{\varrho}\otimes\{\lambda''\}^{\varrho})^{\mathrm{stab}\,\varrho} = \sum_{\beta,\beta',\beta''} j_{\beta,\varrho(\lambda)} j_{\beta',\varrho}(\lambda') j_{\beta'',\varrho}(\lambda'')\left(\prod_{i=1}^m k(\beta^i,\beta'^i,\beta''^i)\right),
$$

where the sum for $\beta$ is over all lists of partitions such that $\beta^i\vdash\kappa_i$ and analogously for $\beta'$ and $\beta''$. □

We can now obtain a second proof for the fact that the hook triple gives an equation for proving lower bounds on the border rank:

**3.35 Corollary.** *Let $\lambda=\lambda'=\lambda''$ be the hook partition with $3k+1$ boxes and $2k+1$ rows. Then $\mathrm{mult}_{(\lambda,\lambda',\lambda'')}(\mathsf{GL}^3_{3k}E_{3k})=0$, where $E_{3k}$ is the $3k$-th unit tensor.*

*Proof.* We use the formula in Theorem 3.33. Since it has no signs, we can assume (for the sake of contradiction) that the formula yields a is positive result and derive conditions on the partitions that are involved in positive summands.

We use a few standard facts about Littlewood-Richardson coefficients, plethysm coefficients, and Kronecker coefficients, each marked with a †.

First observation: $\nu^1=\beta^1$, because of the plethysm $a_{\nu^1}(\beta^1,1)=\mathrm{mult}_{\nu^1}(\underbrace{S^{\beta^1}(\mathrm{Sym}^1 V)}_{=\{\beta^1\}})$.

A multi-LR-coefficients can only be positive if all small partitions are contained in the large partition, i.e., the small Young diagrams are subsets of the large Young diagram (†). In our case, all large partitions are hooks, so all $\nu^i$ are hooks. Thus also $\beta^1$, $\beta'^1$, $\beta''^1$ are hooks.

Let $d$ be the number of boxes. For a hook $\nu^1$ define the *inner leg length* as $\ell(\nu^1) - 1$. For hook triples with inner leg lengths $a_1$, $a_2$, $a_3$, Kronecker positivity requires (†, see e.g. Mercedes Rosas' PhD thesis):

$$2d - a_1 - a_2 - a_3 - 2 \geq 0.$$

Thus not all three $a_1, a_2, a_3$ can be large. Indeed, let $a = \min\{a_1, a_2, a_3\}$, then $2d - 3a - 2 \geq 0$ and thus $a \leq \frac{2d-2}{3}$. In particular this holds for $k(\nu^1, \nu'^1, \nu''^1) = k(\beta^1, \beta'^1, \beta''^1) > 0$. W.l.o.g. $\nu^1$ is the shortest of $\nu^1, \nu'^1, \nu''^1$. Then

$$\ell(\nu^1) - 1 \leq \frac{2|\nu^1| - 2}{3} = \frac{2}{3}|\nu^1| - \frac{2}{3}$$

and thus

$$\ell(\nu^1) \leq \frac{2}{3}|\nu^1| + \frac{1}{3}.$$

All partitions appearing in $\bigotimes^a \operatorname{Sym}^b V$ have at most $a$ rows, as the basis of HWVs is given by semistandard tableaux with content $(b, b, \ldots, b)$. Therefore the positive plethysm coefficients in the formula imply

$$\ell(\nu^i) \leq |\beta^i| = \kappa_i = \frac{\nu^i}{i}$$

Adding up the lengths we obtain

$$\ell(\nu^1) + \cdots + \ell(\nu^\ell) \leq \frac{2}{3}|\nu^1| + \frac{1}{3} + \frac{1}{2}(\underbrace{|\nu^2| + \cdots + |\nu^\ell|}_{=3k+1-|\nu^1|})$$

$$= \frac{2}{3}|\nu^1| + \frac{1}{3} + \frac{3}{2}k + \frac{1}{2} - \frac{1}{2}|\nu^1| = \frac{3}{2}k + \frac{1}{6}|\nu^1| + \frac{5}{6}$$

We now use that for a positive multi-LRC the lengths of the small partitions add up to at least the length of the large partition (†):

$$\ell(\nu^1) + \cdots + \ell(\nu^\ell) \geq \ell(\lambda) = 2k + 1.$$

Therefore

$$\frac{3}{2}k + \frac{5}{6} + \frac{1}{6}|\nu^1| \geq 2k + 1 \Leftrightarrow -\frac{1}{2}k - \frac{1}{6} + \frac{1}{6}|\nu^1| \geq 0 \Leftrightarrow |\nu^1| \geq 3k + 1.$$

Since $|\nu^1| = \kappa_1$, this means that $\varrho^1 = (1^{3k+1})$, but the sum is only over $\varrho^1$ with at most $3k$ rows. □