Fast Multivariate Multipoint Evaluation

Based on joint works with

Vishwas Bhargava, Sumanta Ghosh, Zeyu Guo, Chandra Kanta Mohapatra, Chris Umans

Multipoint evaluation

Input

- An m-variate polynomial f with degree at most (d-1) in each variable over a field **K**, as a list of coefficients
- N points $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbf{K}^{\mathbf{m}}$

Output

- Evaluation of f on $\alpha_1, \alpha_2, \ldots, \alpha_N$

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Input: $(d^m + Nm)$ field elements

Multipoint evaluation

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Naïve algorithm

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For i = 1 to N:

    Evaluate f on α_i

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Roughly (Nmd^m) field operations in total

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Can we do this faster ?

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When $\mathrm{N} = \mathrm{d}^{\mathrm{m}},$ quadratic in the input size

Can we do this faster ?

In particular, is there an algorithm that runs in linear time in the input size ?

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- A very basic and natural algorithmic question in computational algebra
- Many direct and natural applications – fast modular composition, univariate polynomial factorization over finite fields, generating irreducible polynomials, computing minimal polynomials, data structures for polynomial evaluation, ...
- Current fastest algorithms for all these problems go via fast multipoint evaluation

Faster-than-trivial multipoint evaluation

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Multipoint evaluation: the univariate case

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Input is specified via $(N + d)$ field elements

Multipoint evaluation: the univariate case

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For structured set of input points

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For structured set of input points

- when $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbf{K}$ are all roots of unity of order N

Multipoint evaluation: the univariate case

For structured set of input points

- when $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbf{K}$ are all roots of unity of order N
- an algorithm with $(N + d)^{1+o(1)}$ field operations using Fast Fourier Transform

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- **[Borodin-Moenck, 1974]** An algorithm with $(N + d)^{1+o(1)}$ field operations

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- **[Borodin-Moenck, 1974]** An algorithm with $(N+d)^{1+o(1)}$ field operations
- a very clever and neat application of FFT

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- when $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbf{K}$ form a product set, i.e., $\{\alpha_1, \alpha_2, \ldots, \alpha_N\} = S_1 \times S_2 \times \cdots \times S_m$, for $S_i \subseteq \mathbf{K}$

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- an easy nearly linear time algorithm – induction on the number of variables
- uses the univariate case as the base case

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- Nusken-Ziegler designed a slightly faster (though far from linear time) algorithm in 2004
- based on faster rectangular matrix multiplication

The multivariate case: more recent progress

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[Umans, 2008]

A nearly linear time algorithm for multivariate multipoint evaluation when

1. char(**K**) is less than $d^{o(1)}$
2. number of variables (m) is less than $d^{o(1)}$

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Not a polynomial time algorithm, since the running time depends polynomially (and not polylogarithmically) on the field size

Nevertheless, happens to be very useful for one of our results

Multivariate multipoint evaluation

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This is the question that we study in our work and focus of rest of the talk.

Our results

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(degree d is asymptotically growing)

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Our results

Nearly linear time algorithm for multivariate multipoint evaluation over all finite fields, for growing d, and all m

In summary

	Field Size	Characteristic	Number of variables	Algebraic vs non-algebraic
Umans	Finite	char(K) <	m <	Algebraic
Kedlaya-Umans	Finite	All finite fields	m <	Non-algebraic
Bhargava-Ghosh-K-Mohapatra	Not-too-large	char(K) <	No constraint	Algebraic
Bhargava-Ghosh-Guo-K-Umans	Finite	All finite fields	No constraint	Non-algebraic

An outline of the algorithm

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Theorem

A nearly linear time algorithm for multivariate multipoint evaluation when

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- Preprocessing phase: independent of the evaluation points $\alpha_1, \alpha_2, \ldots, \alpha_N$

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Two phases of the algorithm

- Preprocessing phase: independent of the evaluation points $\alpha_1, \alpha_2, \ldots, \alpha_N$
- Local computation phase: depend on $\alpha_1, \alpha_2, \ldots, \alpha_N$, and earlier computation

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2. Evaluate f on all points of S

An outline of the algorithm

Local computation

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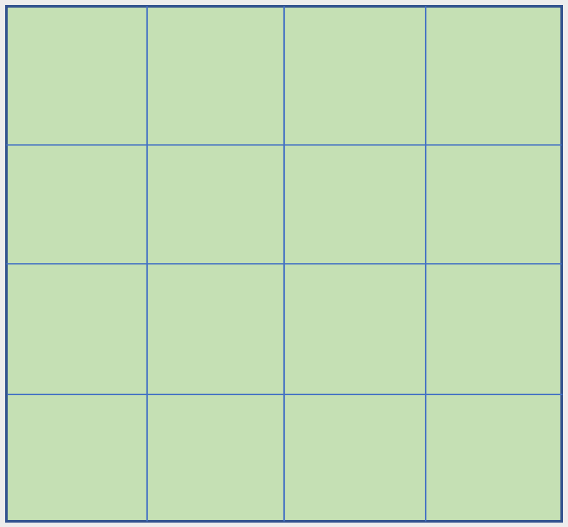
Local computation

- have an $\alpha \in \mathbf{K}^m$, want to compute $f(\alpha)$ fast, using info from the previous step

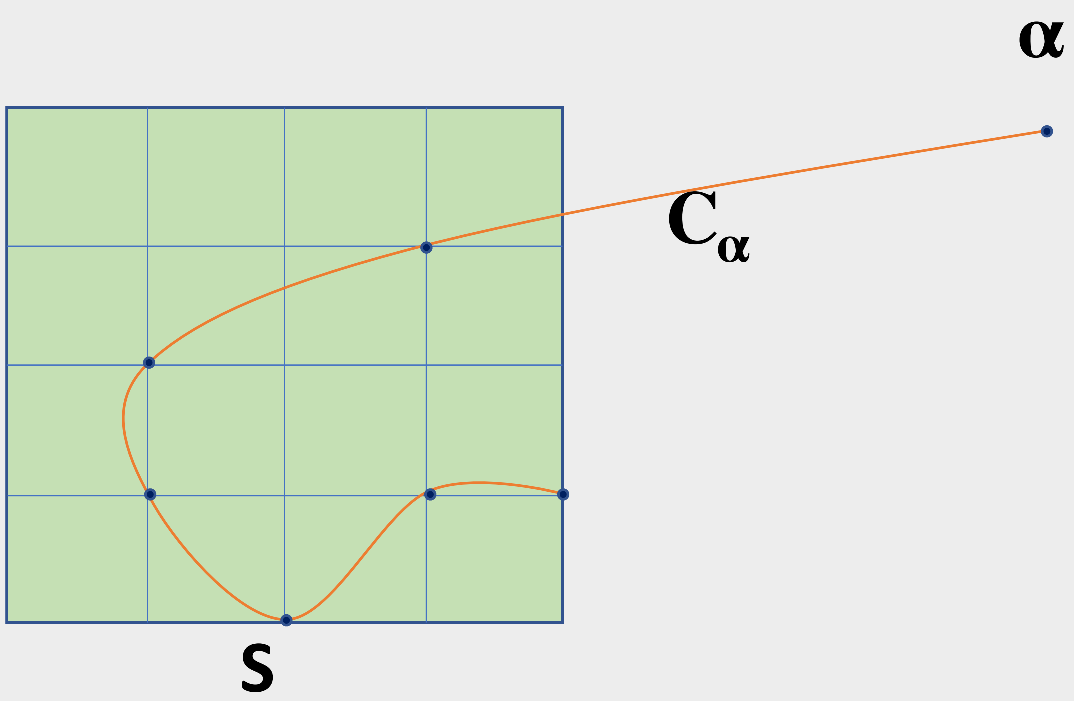
An outline of the algorithm

α



S

An outline of the algorithm



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Local computation

- have an $\alpha \in \mathbf{K}^m$, want to compute $f(\alpha)$ fast, using info from the previous step
- let $C_\alpha(y) = (r_{\alpha,1}(y), r_{\alpha,2}(y), \ldots, r_{\alpha,m}(y))$ be the low degree curve through α, with large intersection with S; each $r_{\alpha,i}(y)$ is a low degree polynomial

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- if we can efficiently get our hands on g, we can set t = u, to get $f(\alpha)$

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- once, we have g, can recover $g(u) = f(\alpha)$

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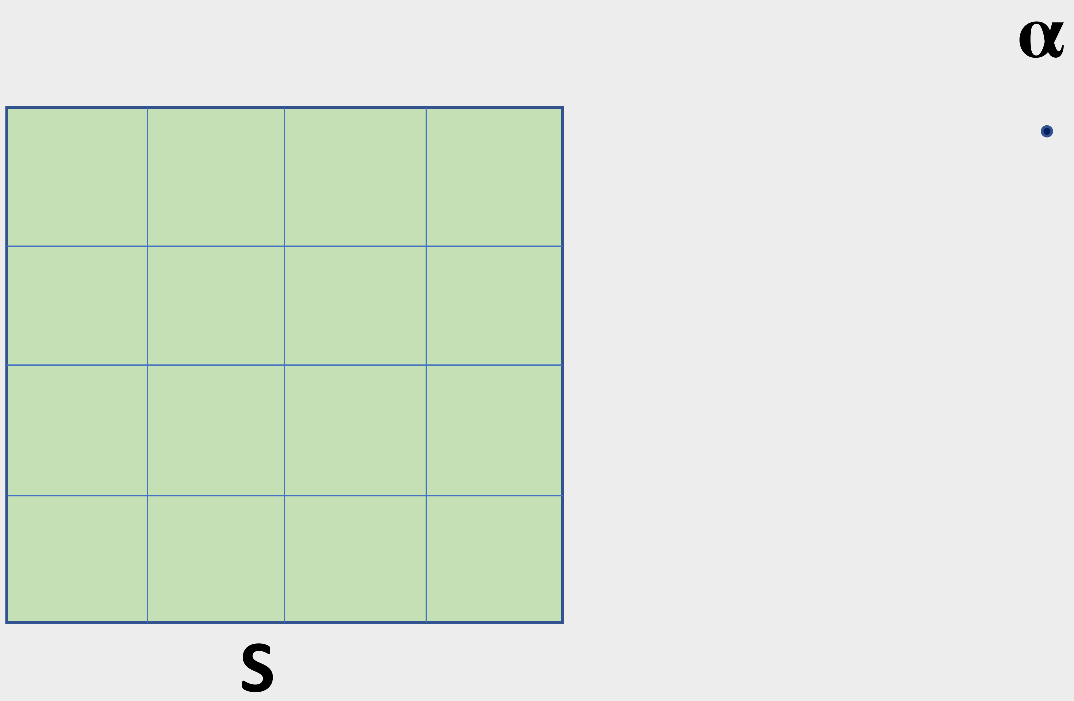

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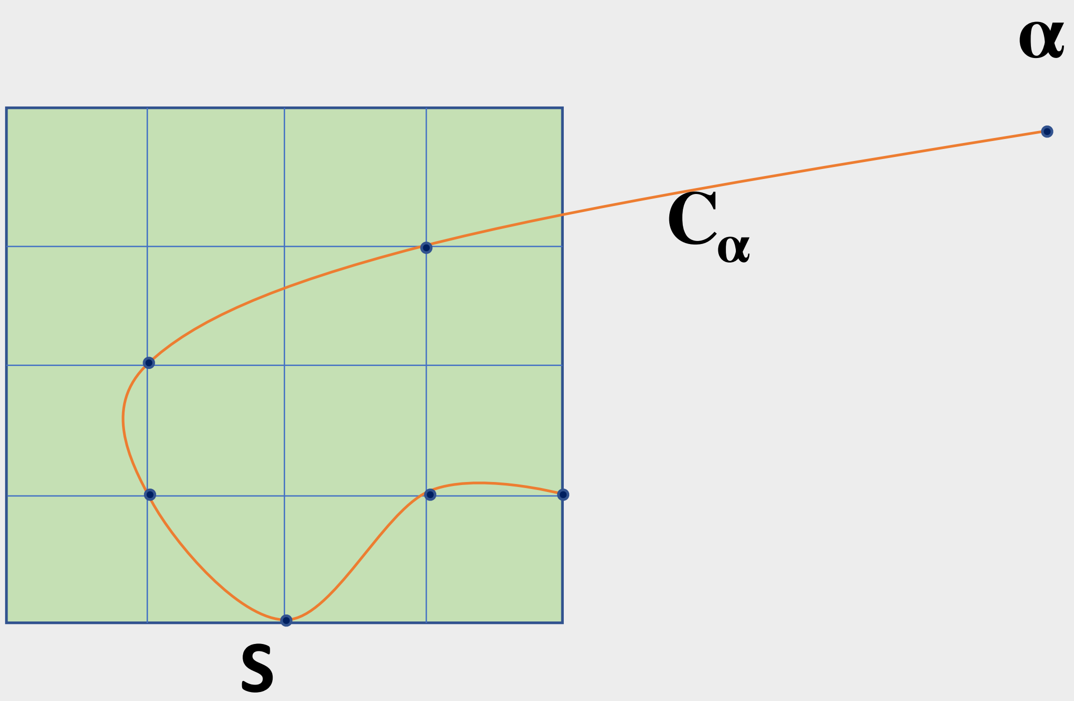

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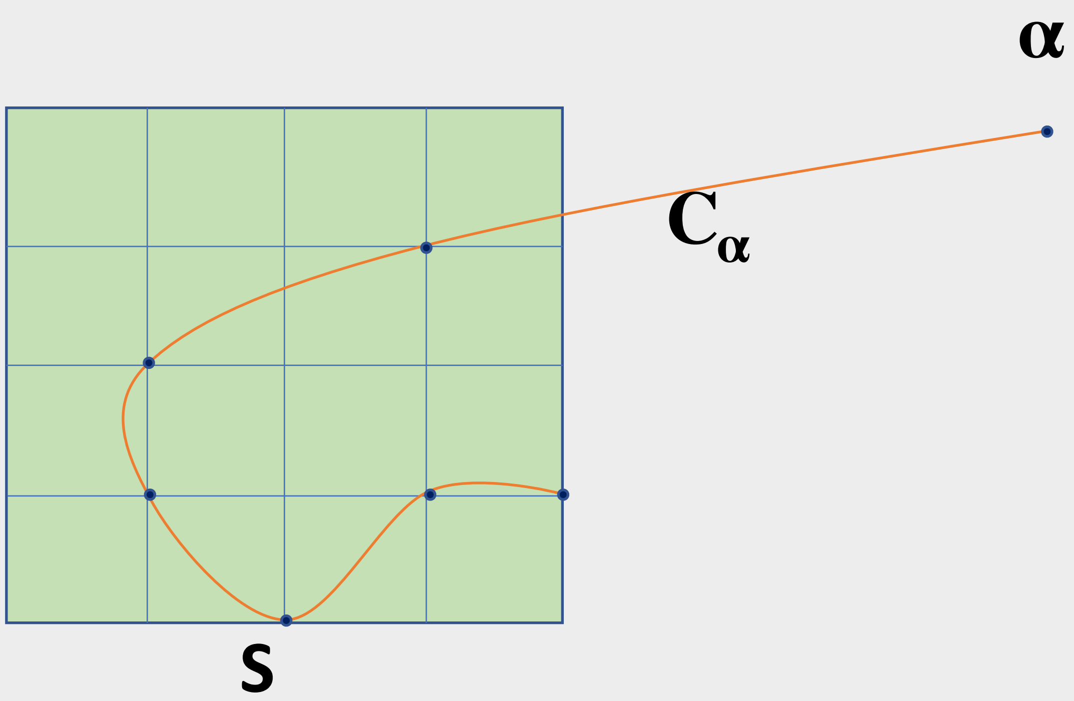

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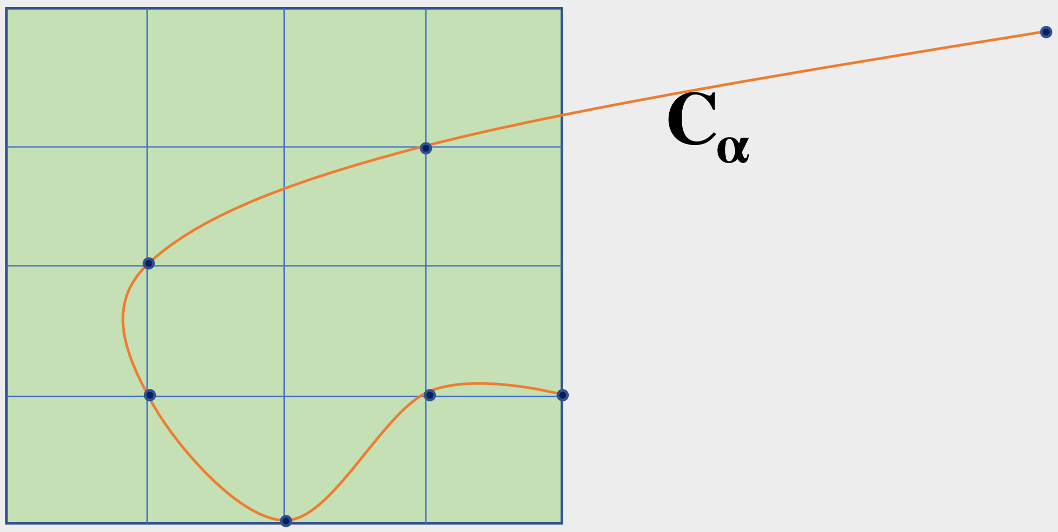

Want

$$|\mathbf{C}_\alpha \cap S| > \mathbf{deg}(\mathbf{C}_\alpha) \cdot \mathbf{dm}$$

- $|S| < \left(\mathrm{pdm} \cdot \log_{\mathbf{p}} |\mathbf{K}|\right)^{\mathbf{m}}$
- $\deg(C_\alpha) < \log_{\mathbf{p}} |\mathbf{K}|$
- $|C_\alpha \cap S| > \log_{\mathbf{p}} |\mathbf{K}| \cdot \mathrm{dm} > \deg(C_\alpha) \cdot \mathrm{dm}$

The mysterious set S

- ends up being a vector space over a subfield of appropriate size
- requires the characteristic of the underlying field to be small, else, unclear if such a set exists
- curve property follows from structure of field extensions

Running time

Running time

running time of the first phase – nearly linear in $(d^m + |S|) \sim \left(pdm \cdot \log_{\mathbf{p}} |\mathbf{K}|\right)^{\mathbf{m}}$

N iterations of univariate polynomial interpolation for degree $\log_{\mathbf{p}} |\mathbf{K}| \cdot dm$ + finding the curves at each input

overall running time : $(N + \left(pdm \cdot \log_{\mathbf{p}} |\mathbf{K}|\right)^{\mathbf{m}}) \cdot poly(\log_{\mathbf{p}} |\mathbf{K}| \cdot dm)$

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- to continue the local decoding step, will need to ensure that we have sufficient information for univariate interpolation along the curve at each point
- here, we work with a smaller set S
- leads to reduced intersection between the curves and the set S
- to compensate, need stronger preprocessing phase, and a more complicated local computation step

A few more ideas

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- method of multiplicities
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- this additional information lets us proceed with a smaller set S

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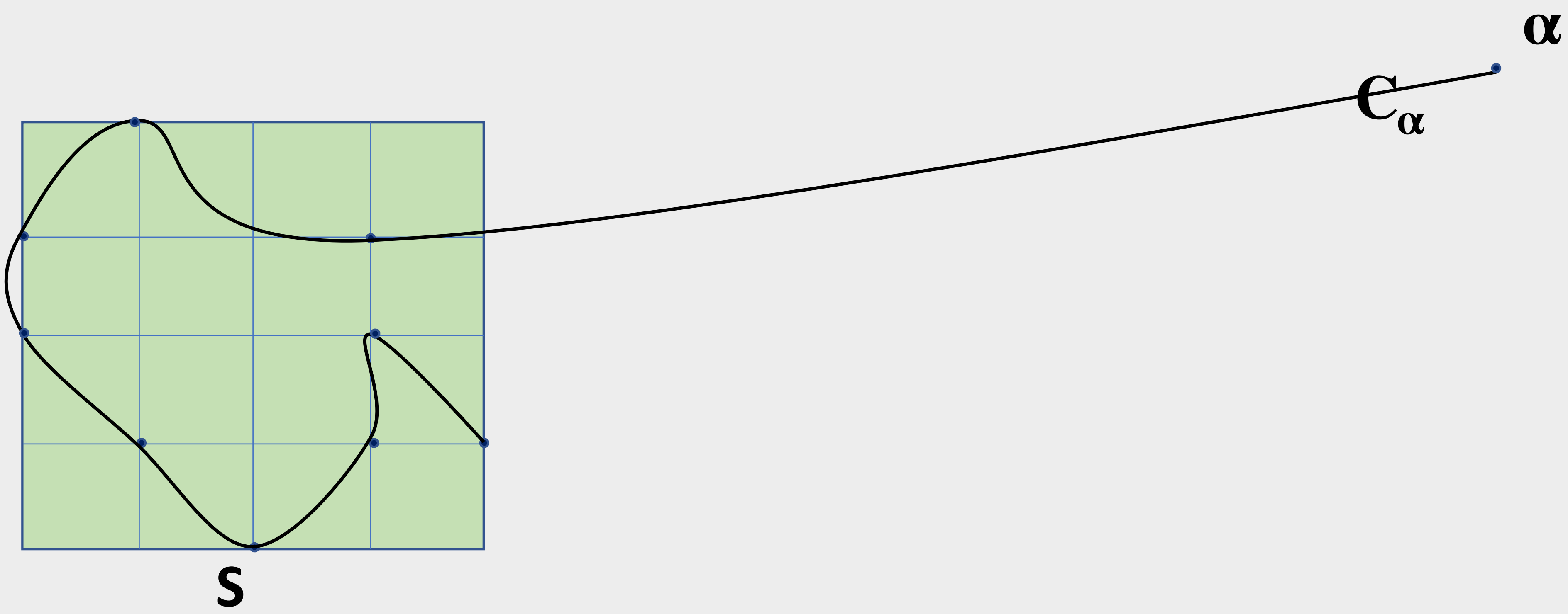
- then, use this additional info, together with values of f on S to do interpolation

The final inaccurate picture



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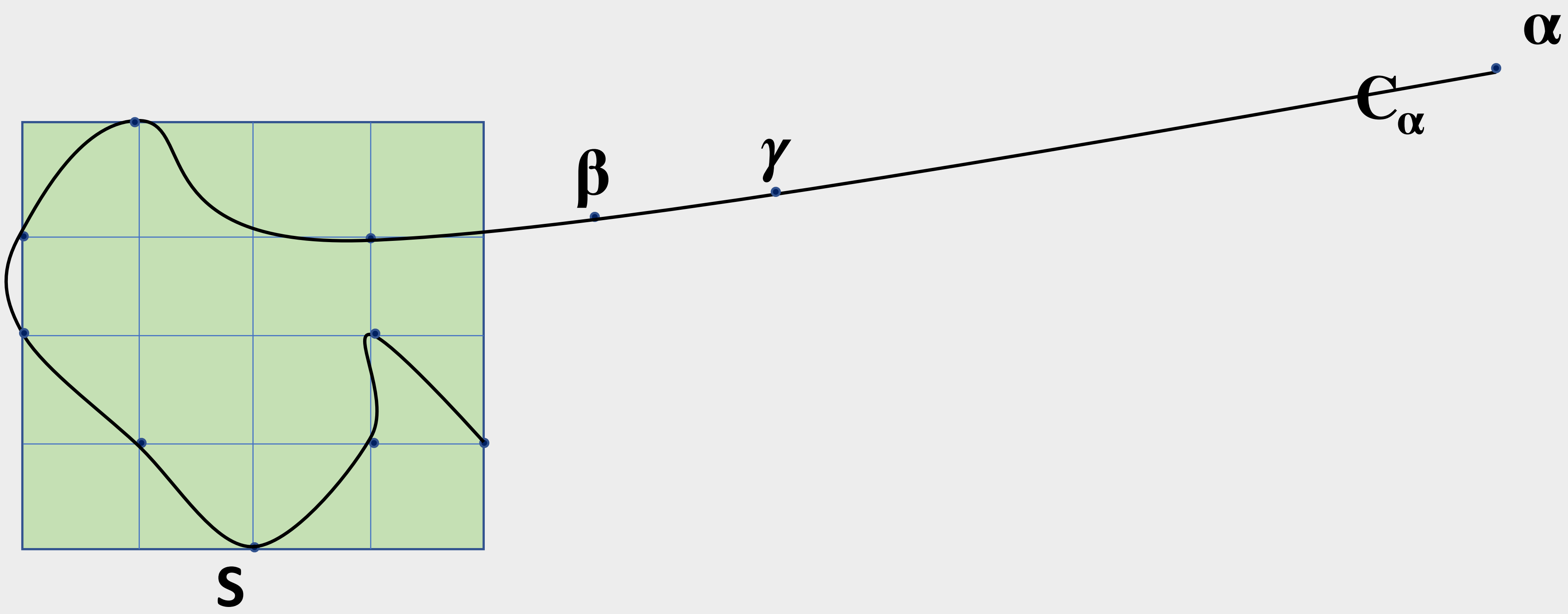

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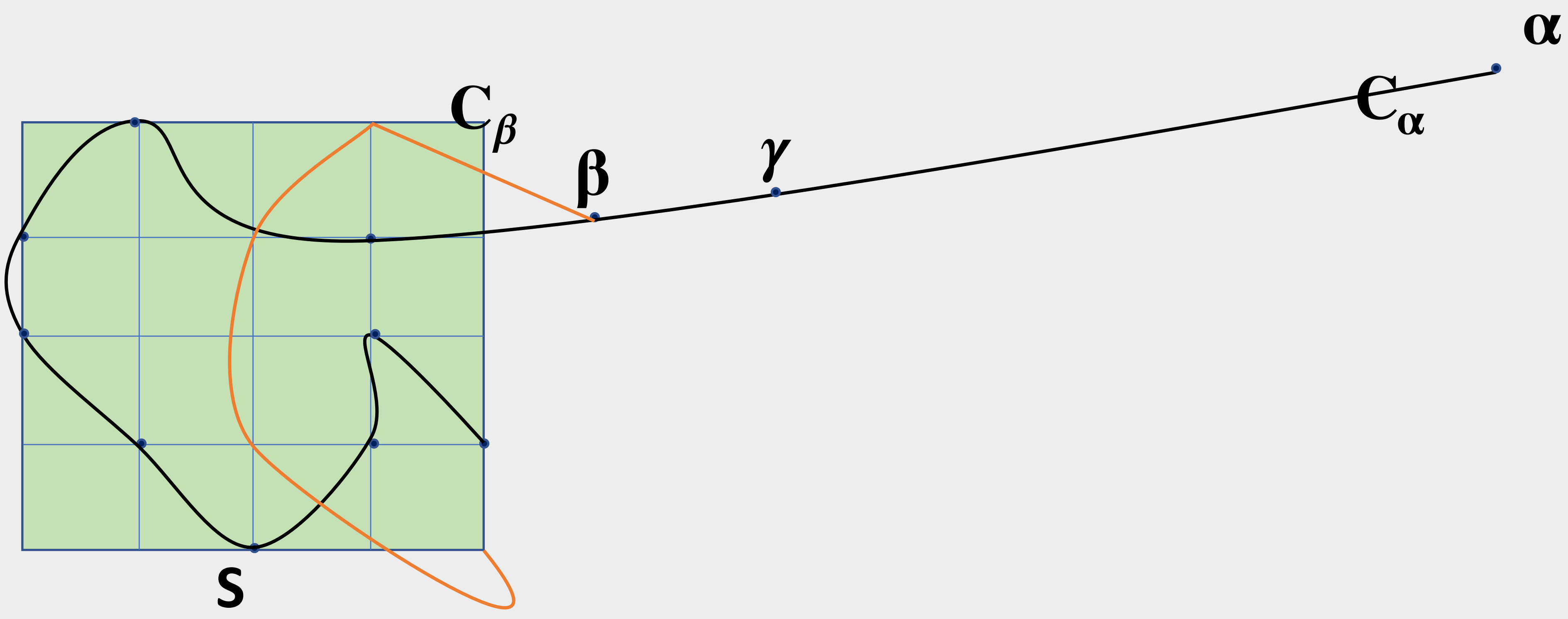

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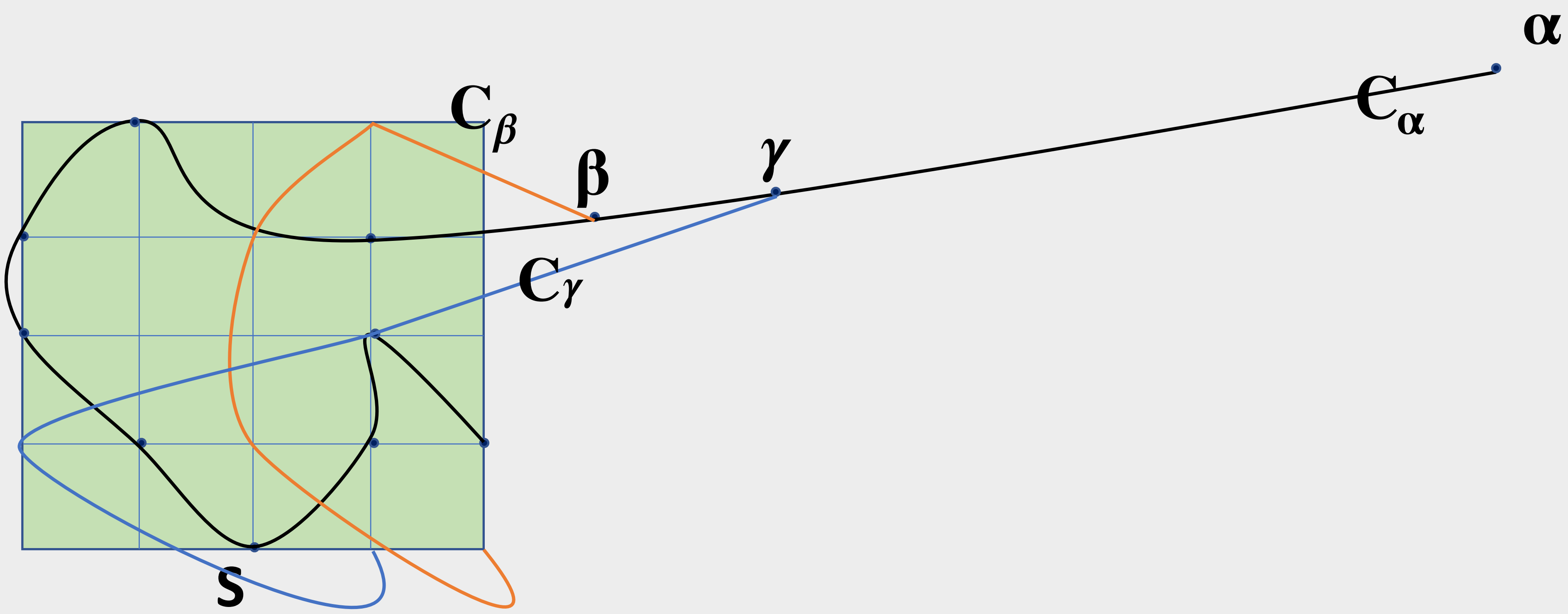

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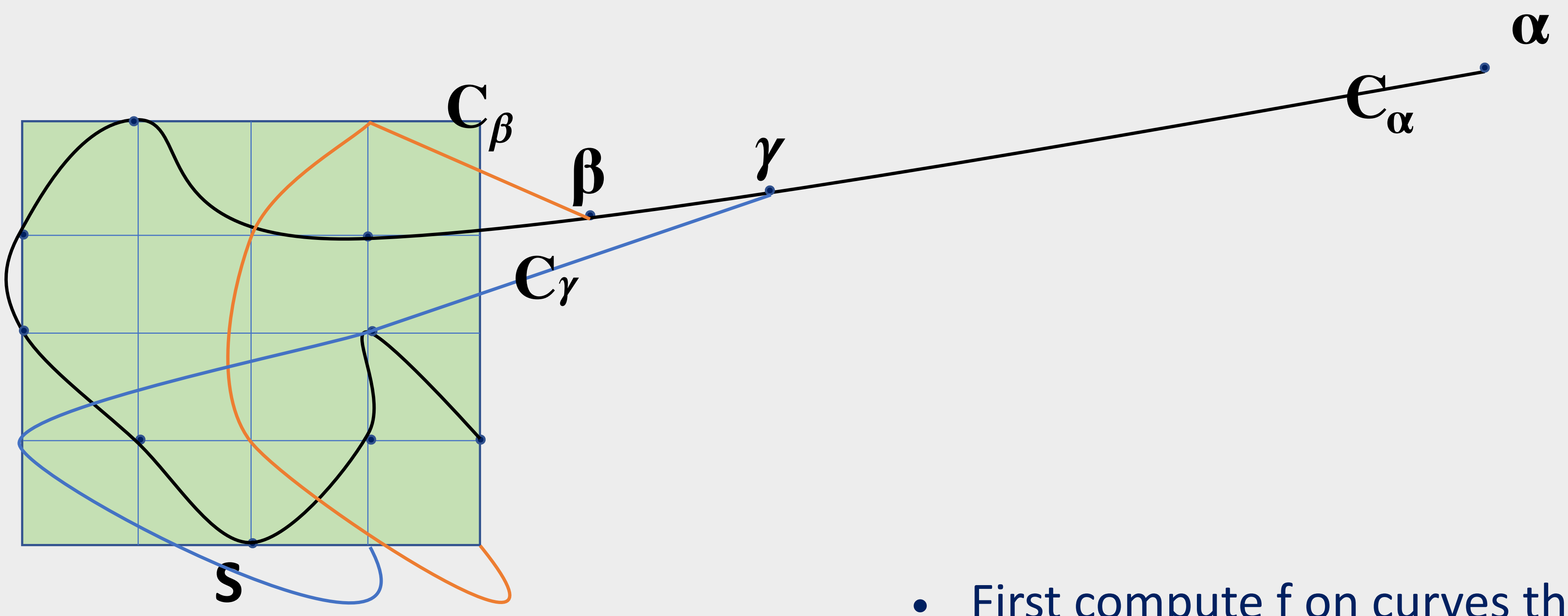

- First compute f on curves through simpler points β, γ using the previous algorithm
- Then, use the values of f on S, and curves through β, γ to compute f on C_α

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- crucially uses a result of Bombieri-Vinogradov about the density of primes in an arithmetic progression
- essentially, both improve some of the bottlenecks in Kedlaya-Umans using ideas from the small characteristic case and BKW19 in slightly different ways

Open Questions

- An algebraic algorithm over finite fields ?
- An algorithm (or an algebraic circuit) over infinite fields (complex numbers) ?
- More applications ?
- What about faster algorithms for other related problems ? e.g. multivariate interpolation ?
- What about the case of constant d ? e.g. multilinear polynomials ?

Thank You!