Schur Polynomials do not have small formulas if the Determinant doesn't

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Joint work with

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WACT 2023

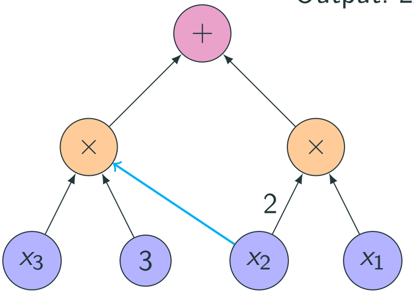
University of Warwick

Talk Outline

(a) Preliminaries
(b) Introduction and prior work
(c) Main results
(d) Proof sketch
(e) Open questions

Algebraic circuit

Output: $2x_1x_2 + 3x_2x_3$



Algebraic Formula

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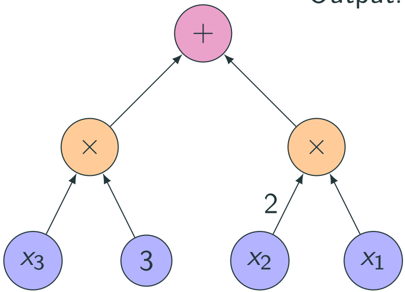
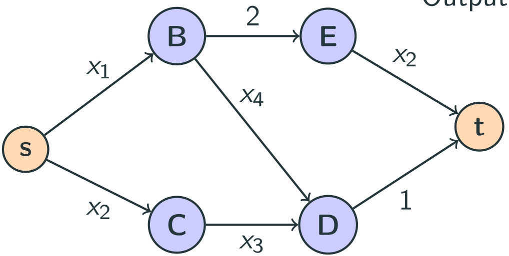


Algebraic Branching Program



Output: $2x_1x_2 + x_1x_4 + x_2x_3$

Introduction: Symmetric Polynomials

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$f(x_1, x_2) = x_1 + x_2$ is symmetric but $f(x_1, x_2) = {x_1}^2 + x_2$ is not.

Introduction: Symmetric Polynomials

The elementary symmetric polynomial(e_d) is the sum of all multilinear monomials of degree exactly d.

$$e_d = \sum_{i_1 < i_2 < \ldots < i_d} x_{i_1} x_{i_2} \ldots x_{i_d}$$

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Introduction

[Lipton-Regan '09] complexity of symmetric polynomials

$\Downarrow$?

complexity of polynomials in general

Introduction

Fundamental theorem of symmetric polynomials

For any $f_{\text{sym}} \in \mathbb{C}[x_1, x_2 \ldots x_n]$, there exists a unique $f \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ such that $f_{\text{sym}} = f(e_1, e_2 \ldots, e_n)$

$$e_d \stackrel{\text{def}}{=\!=} \text{elementary symmetric poly of deg } d$$

Assumption Complex field

How the complexity of f and f_{sym} are related?

Introduction

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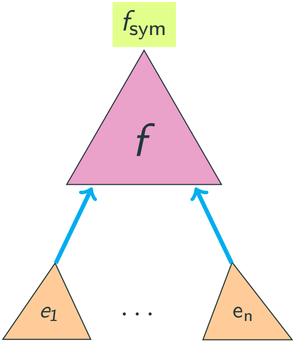

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[Bläser-Jindal '18] answers this affirmatively.

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only for circuits

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[Bläser-Jindal '18] $C(f) \leq \mathcal{O}(d^2 C(f_{\text{sym}}) + d^2 n^2)$ $\quad (d \stackrel{\text{def}}{=\!=} \deg(f))$

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Can we prove a similar statement for the ABPs(Formulas)?

Our result for Formulas

[Bläser-Jindal '18] For any polynomial $f \in \mathbb{C}[\mathbf{x}]$ of deg d where $f_{\text{sym}} = f(e_1, \ldots, e_n)$,

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Our result for Formulas

[Bläser-Jindal '18] For any polynomial $f \in \mathbb{C}[\mathbf{x}]$ of deg d where $f_{\text{sym}} = f(e_1, \ldots, e_n)$,

$$C(f) \leq \mathcal{O}(d^2 C(f_{sym}) + d^2 n^2)$$

[This work] There exists $\boldsymbol{b} \in \mathbb{C}^n$, s.t. for any homogeneous polynomial $f \in \mathbb{C}[\mathbf{x}]$ of deg d, if $f_{\text{sym}} = f(e_1 - b_1, \ldots, e_n - b_n)$ then,

$$L(f) \leq \mathcal{O}(L(f_{\text{sym}})^2 n)$$

$L(f) \stackrel{\text{def}}{=\!=}$ formula size of 'f'

Generalized Vandermonde Matrix

Principal Vandermonde Matrix

$$V_n = \begin{pmatrix} x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \dots & x_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \\ 1 & 1 & \dots & 1 \end{pmatrix}_{n\times n}$$

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$$\det(V_n) = \prod_{i<j} (x_i - x_j)$$

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$$\mathrm{GV}_n^{\boldsymbol{t}} = \begin{pmatrix} x_1^{t_1} & x_2^{t_1} & \dots & x_n^{t_1} \\ x_1^{t_2} & x_2^{t_2} & \dots & x_n^{t_2} \\ \vdots & \vdots & \ddots & \vdots \\ \\ x_1^{t_n} & x_2^{t_n} & \dots & x_n^{t_n} \end{pmatrix}_{n \times n}$$

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[This work] There are G.V. matrices whose Det. doesn't have a small small formula if the symbolic Det. does not.

Schur Polynomial

Schur Polynomial of degree d over its partition $\boldsymbol{\lambda}$ is defined as

$$S_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_n) = \frac{\det\left(\mathrm{GV}_n^{\boldsymbol{\lambda}+\boldsymbol{\delta}}\right)}{\det(V_n)}$$

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$$t_i \leftarrow \lambda_i + n - i$$

Formula complexity of S_λ

$$S_{\boldsymbol{\lambda}}(\mathbf{x}) = \frac{\det\left(\mathsf{GV}_n^{\boldsymbol{\lambda}+\boldsymbol{\delta}}\right)}{\det(V_n)}$$

Q. What is the formula complexity of Schur polynomials?

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There are G.V.Ds which do not have small formulas if the symbolic Determinant does not.

Proof idea

Input:

1. $g(q_1, q_2 \ldots, q_k)$ where $q_i \in \mathbb{C}[x_1, x_2 \ldots x_n]$ and q_i's are algebraically independent.
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Our technique: For some $\mathbf{a} \in \mathbb{C}^n$

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$$g(q_1(\mathbf{a}+\mathbf{x}), q_2(\mathbf{a}+\mathbf{x}), \ldots, q_k(\mathbf{a}+\mathbf{x}))$$

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$$q_i(\mathbf{a} + \mathbf{x}) = \text{linear component}(\mathcal{L}_i) + \text{higher degree components}(\mathcal{H}_i)$$

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$\downarrow$ degree d component

$$g(\mathcal{L}_1(\mathbf{x}), \mathcal{L}_2(\mathbf{x}), \ldots \mathcal{L}_k(\mathbf{x}))$$

Proof idea

Input:

1. $g(q_1, q_2 \ldots, q_k)$ where $q_i \in \mathbb{C}[x_1, x_2 \ldots x_n]$ and q_i's are algebraically independent.
2. g is a homogeneous poly of degree d where $\{q_1, q_2 \ldots q_k\}$ satisfies some special property.

Output: Find $g(z_1, z_2 \ldots z_k)$ efficiently.

Our technique:

For some $\mathbf{a} \in \mathbb{C}^n$

$\Big\downarrow$ Taylor expansion

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$$g(z_1, z_2 \ldots z_k)$$

Key lemma

1. g is a homogeneous poly of degree d.
2. $g(q_1, q_2 \ldots, q_k)$ has a small formula, where $q_i \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ and q_i's are algebraically independent.

There exists a point '**a**' s.t.

i $q_i(\mathbf{a}) = 0$ for all i.

ii The rank of the Jacobian matrix of $q_1, q_2, \ldots, q_k$ when evaluated at '**a**' is equal to its symbolic rank.

$$\Downarrow$$

$g(z_1, z_2 \ldots z_k)$ has a small formula.

Summary of results

Theorem

$\exists \boldsymbol{b} \in \mathbb{C}^n$ s.t. for any homogeneous polynomial $f \in \mathbb{C}[\mathbf{x}]$ of deg d, if $f_{sym} = f(e_1 - b_1, \ldots, e_n - b_n)$ then,

$$L(f) \leq \mathcal{O}(L(f_{sym})^2 n)$$

$$L(f) \stackrel{\text{def}}{=\!=} \text{formula size of } f$$

Theorem

There exists a $\boldsymbol{\lambda}$ s.t. $S_{\boldsymbol{\lambda}}$ does not have a small formula unless the Determinant has.

Theorem

There are Generalized Vandermonde determinants which do not have small formulas if the determinant does not.

Open questions

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1. Can we eliminate the homogeneity constraint on g?
2. Can we eliminate the special properties?
3. Can we prove a Bläser & Jindal kind of statement for formulas and ABPs in general?

Thank you!