# Border Complexity of Symbolic Determinant with Rank 1 constraint

## WACT 2023

(University of Warwick)

Joint work with Abhranil Chatterjee, Sumanta Ghosh and Rohit Gurjar

# Symbolic Determinant

$$X = \{x_1, x_2, \dots x_n\} \quad , \ \mathbb{F}$$

$$\begin{array}{cccccl} & & & \text{jth column} & & \\ A = & \left(\begin{array}{cccc} l_{11}(X) & \cdots & \cdot & l_{1r}(X) \\ \vdots & \ddots & & \\ & & l_{ij}(X) & \\ l_{r1}(X) & \cdots & & l_{rr}(X) \end{array}\right) & & & & \text{ith row} \end{array} = A_0 + \sum_{i=1}^{n} A_i x_i$$

$$A_i \in \mathbb{F}^{r \times r}$$

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$$\det(A) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} l_{i\pi(i)}$$

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Mulmuley and Sohoni proposed Geometric Complexity Theory as a possible approach to prove the conjecture

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$$g = \lim_{\epsilon \to 0} g_\epsilon = 3xy^2 \in \overline{WR_2} \text{ but } \notin WR_2$$

# $\overline{\text{VBP}}$ vs VNP Conjecture

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$\overline{\text{VBP}} \stackrel{?}{=} \text{VBP}$ Major open question of GCT

(Results are known for some subclasses)

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ABPs with edge labels in $\mathbb{R}[\epsilon, \epsilon^{-1}]_+[X]$ $\quad$ $\mathbb{R}_+[X]$

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This Work $\rightarrow$ Symbolic Determinant with Rank-1 constraint

# Rank 1 Constraint

Def:

$$\mathrm{DET1}_{k,n} = \left\{ \det\left(A_0 + \sum_{i=1}^{n} A_i x_i\right) : A_i \in \mathbb{C}^{k \times k}, \ rk(A_i) = 1 \ \forall i \in [n] \right\}$$

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* $\mathrm{VBP} \subseteq \overline{\mathrm{orbit}(\mathrm{DET1})}$

# Our Result

The class of polynomials computed by determinant of symbolic matrix with rank 1 constraint is $\mathbb{C}$-closed under approximation.

Thm →

$$\overline{\mathrm{DET1}} = \mathrm{DET1}$$

Moreover, $\overline{\mathrm{DET1}_{k,n}} = \mathrm{DET1}_{k,n}$ if $A_0 = 0$

otherwise $\overline{\mathrm{DET1}_{k,n}} \subseteq \mathrm{DET1}_{k+n,n}$

# Another Form of DET1

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$$U = \begin{bmatrix} \uparrow & & \uparrow & & \uparrow \\ \vec{u}_1 & \dots & \vec{u}_i & \dots & \vec{u}_n \\ \downarrow & & \downarrow & & \downarrow \end{bmatrix} \quad , \quad V = \begin{bmatrix} \uparrow & & \uparrow & & \uparrow \\ \vec{v}_1 & \dots & \vec{v}_i & \dots & \vec{v}_n \\ \downarrow & & \downarrow & & \downarrow \end{bmatrix}$$

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**Proof:**

$$A_i x_i = \begin{bmatrix} \uparrow & & \uparrow & & \uparrow \\ \vec{0} & \dots & \vec{u}_i & \dots & \vec{0} \\ \downarrow & & \downarrow & & \downarrow \end{bmatrix} \underbrace{\begin{bmatrix} x_1 & & & & \\ & x_2 & & & \\ & & \ddots & & \\ & & & x_i & \\ & & & & \ddots \\ & & & & & x_n \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \leftarrow & \vec{v}_1^{\,T} & \rightarrow \\ & \vdots & \\ \leftarrow & \vec{v}_i^{\,T} & \rightarrow \\ & \vdots & \\ \leftarrow & \vec{v}_n^{\,T} & \rightarrow \end{bmatrix}}_{V^T}$$

Hence, $\sum_{i=1}^{n} A_i x_i = UXV^T$

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**Thm** (Restated)

> Given $U, V \in \mathbb{C}(\epsilon)^{k \times n}$ s.t. $\lim_{\epsilon \to 0} \det(U_S) . \det(V_S)$ is defined $\forall S \in \binom{[n]}{k}$,
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> then $\exists\, \hat{U}, \hat{V} \in \mathbb{C}^{k \times n}$ s.t. $\forall S \in \binom{[n]}{k}$
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Equivalently,

$$GV^2 = \left\{ \left(\det(U_S)\det(V_S)\right)_{S \in \binom{[n]}{k}} : U, V \in \mathbb{C}^{k\times n} \right\} \text{ is euclidean closed.}$$

# Grassmanian Variety

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# Grassmanian Variety

$\left\{ \left(\det(U_S)\right)_{S \in \binom{[n]}{k}} : U \in \mathbb{C}^{k \times n} \right\}$ is a variety characterized by Grassmannian-Plücker relations.

Equivalently,

Given $U \in \mathbb{C}(\epsilon)^{k \times n}$ with $\lim_{\epsilon \to 0} \det(U_S)$ is defined $\forall S \in \binom{[n]}{k}$, then $\exists\, \hat{U} \in \mathbb{C}^{k \times n}$ with $\lim_{\epsilon \to 0} \det(U_S) = \det(\hat{U}_S)$

Does this directly imply our result? No

**Example**

$$U = \begin{pmatrix} 0 & 1 & 0 & \epsilon \\ 1/\epsilon & 0 & 1 & \epsilon \end{pmatrix} \qquad V = \begin{pmatrix} 1 & \epsilon & 0 & 0 \\ 0 & 0 & 1/\epsilon & 1 \end{pmatrix}$$

| $S$ | $\{1,2\}$ | $\{1,3\}$ | $\{1,4\}$ | $\{2,3\}$ | $\{2,4\}$ | $\{3,4\}$ |
|---|---|---|---|---|---|---|
| $\det(U_S)$ | $-1/\epsilon$ | $0$ | $-1$ | $1$ | $\epsilon$ | $-\epsilon$ |
| $\det(V_S)$ | $0$ | $1/\epsilon$ | $1$ | $1$ | $\epsilon$ | $0$ |
| $\det(U_S)\cdot\det(V_S)$ | $0$ | $0$ | $-1$ | $1$ | $\epsilon^2$ | $0$ |

# Proof Idea

Using results of Murota'96, we show

**Lemma:** Given $U, V \in \mathbb{C}(\epsilon)^{k\times n}$ s.t. $\lim_{\epsilon\to 0} \det(U_S)\det(V_S)$ is defined $\forall S \in \binom{[n]}{k}$,

then $\exists\, \hat{U}, \hat{V} \in \mathbb{C}(\epsilon)^{k\times n}$ s.t. $\forall S \in \binom{[n]}{k}$,

$\lim_{\epsilon\to 0} \det(\hat{U}_S)$ and $\lim_{\epsilon\to 0} \det(\hat{V}_S)$ are defined and $\det(\hat{U}_S)\det(\hat{V}_S) = \det(U_S) \cdot \det(V_S)$

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**Example:** (Revisit)

$$\hat{U} = \begin{pmatrix} 1/\epsilon^2 & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 & 0 & \epsilon \\ 1/\epsilon & 0 & 1 & \epsilon \end{pmatrix}}_{U} \begin{pmatrix} \epsilon & & & \\ & \epsilon^2 & & \\ & & 1 & \\ & & & \epsilon \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & \epsilon^2 \end{pmatrix}$$

$$\hat{V} = \begin{pmatrix} \epsilon^2 & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & \epsilon & 0 & 0 \\ 0 & 0 & 1/\epsilon & 1 \end{pmatrix}}_{V} \begin{pmatrix} 1/\epsilon & & & \\ & 1/\epsilon^2 & & \\ & & 1 & \\ & & & 1/\epsilon \end{pmatrix} = \begin{pmatrix} \epsilon & \epsilon & 0 & 0 \\ 0 & 0 & 1/\epsilon & 1/\epsilon \end{pmatrix}$$

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There exist $c \in \mathbb{Z}$ and $\alpha \in \mathbb{Z}^n$ s.t. multiplying ith column of $U/V$ by $\epsilon^{\alpha_i}/\epsilon^{-\alpha_i}$ and any row of $U/V$ by $\epsilon^c/\epsilon^{-c}$, we get $\hat{U}, \hat{V}$ s.t. $\lim_{\epsilon\to 0} \det(\hat{U}_S)$ and $\lim_{\epsilon\to 0} (\hat{V}_S)$ are defined.

# Future Work

* Find set of Characterizing equations for DET1 class.

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* For $c \in \mathbb{Z}^+$, we can define

$$DETc_{k,n} = \{ \det(A_0 + \sum A_i x_i) : A_i \in \mathbb{C}^{k \times n}, rk(A_i) \leq c \}$$

and $DETc_n \rightarrow k$ is poly$(n)$.

Is $\overline{DETc_n} = DETc_n$ ?

Thank You!