Rabbits Approximate, Cows compute Exactly!

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based on joint work with

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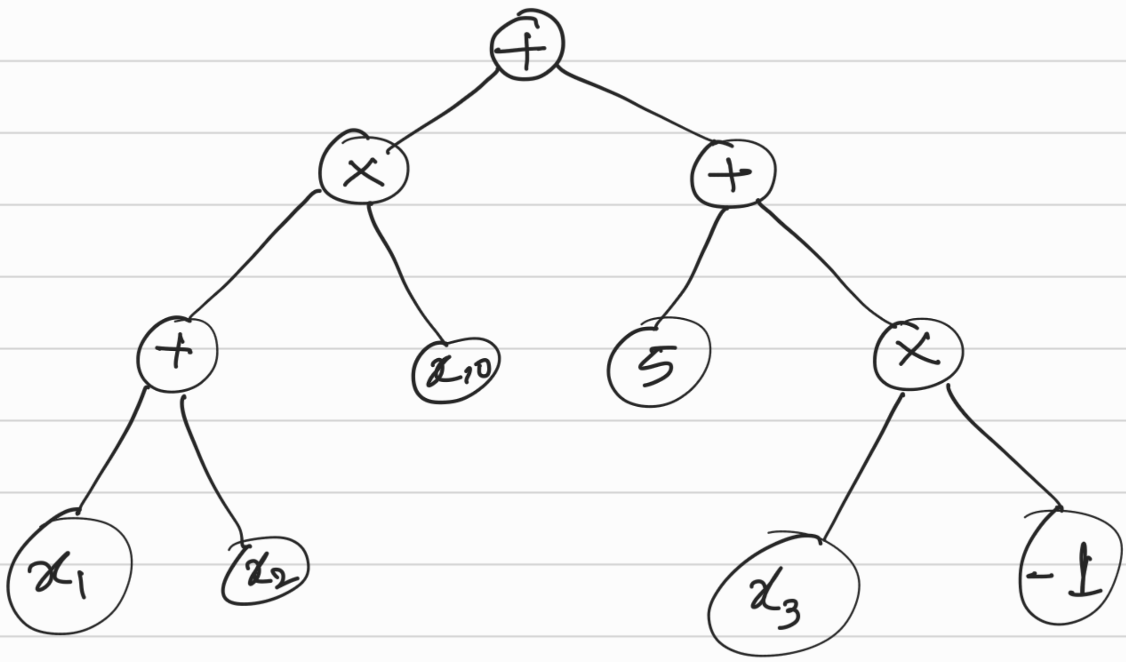
Question

* What is the simplest class of matrices whose determinants can exactly & efficiently simulate algebraic formulas?

VF = polynomial families that can be computed by formulas of polynomial size.



Size = # leaves

depth = length of a longest path from root to a leaf

$$\mathrm{Det}_{n\times n} = \det\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ x_{n1} & \cdots & \cdots & x_{nn} \end{pmatrix}$$

* Universal computational model

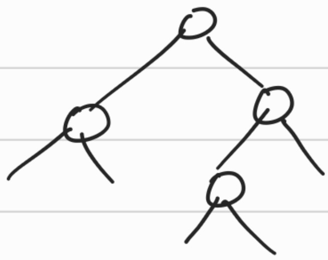
* VDet = polynomial families that can be expressed as det of a matrix of polynomially bounded dimension

e.g.

$$x_1 x_2 x_3 \cdots x_n = \det\begin{pmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots & 0 \\ 0 & 0 & x_3 & \cdots & 0 \\ \vdots & & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & x_n \end{pmatrix}$$

* [Valiant '79] $\mathrm{VF} \subseteq \mathrm{VDet}$

In particular,



$\rightsquigarrow \det \begin{pmatrix} \quad \end{pmatrix}_{(s+2)\times(s+2)}$

Formula of size s

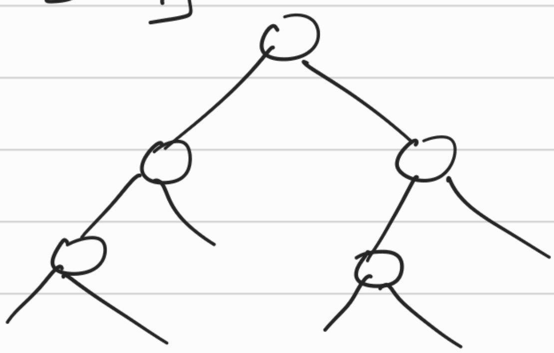
* What about the converse?

i.e. Is $\mathrm{VDet} \subseteq \mathrm{VF}$?

OPEN!

* [Csanky '76, Berkowitz '84]

$\det_{n\times n} \rightsquigarrow$



size $n^{O(\log n)}$

Rephrasing the question

Is there a class of matrices for which the determinant family is computationally equivalent to algebraic formulas ?

* **Projections:**

$$f(x_1, \dots, x_n) \leq g(y_1, \dots, y_m)$$

iff $f(x_1, \dots, x_n) = g(z_1, \dots, z_m)$

s.t. $z_i \in \{x_1, \dots, x_n\} \cup \{\text{constants}\}$.

E.g.:

$$\det \begin{pmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_n \end{pmatrix} = x_1 \cdot x_2 \cdots x_n$$

Can not even compute $x+y$.

$$
* \quad \det \begin{pmatrix} x_{11} & x_{12} & & & & \\ x_{21} & x_{22} & x_{23} & & 0 & \\ & x_{32} & x_{33} & x_{34} & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & & & x_{n-1,n} \\ & & & & x_{n,n-1} & x_{nn} \end{pmatrix}
$$

Tridiagonal matrices.

* [Allender - Wang '16] such families can not even compute

$$x_1x_2 + x_3x_4 + \cdots\cdots + x_{15}x_{16}.$$

* [Bringmann-Ikenmeyer-Zuiddam '18]

Such families can approximate formulas!

In other words,

$$\overline{VF} = \overline{\det(\text{tridiagonal})}$$

Furthermore,

$$= \overline{\det \begin{pmatrix} x_1 & -1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & 1 & x_n \end{pmatrix}}$$

Continuant polynomials.

* Tetradiagonal matrices.

$$\det \begin{pmatrix} x_{11} & x_{12} & x_{13} & & & 0 \\ x_{21} & x_{22} & x_{23} & x_{24} & & \\ & x_{32} & x_{33} & x_{34} & x_{35} & \\ & & \ddots & \ddots & \ddots & \ddots \\ & 0 & & \ddots & \ddots & \ddots \end{pmatrix}_{n \times n}$$

It's not even clear that it is universal.

* [Ben-Or – Cleve '92] width-3 BP is equivalent to formulas.

$$\Rightarrow \det \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & & & \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & & \\ & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} & x_{37} & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}_{n \times n}$$

diagonals $= 6$.

Our Result

* determinant family of tetradiagonal matrices is complete for VF.

This is all fine .... BUT

What does this got to do

with Rabbits & Cows ?!

Continuant Polynomial.

* $$F_n := \det \begin{pmatrix} x_1 & -1 & & & \\ 1 & x_2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & x_n \end{pmatrix}$$

Then,

$$F_n = x_n \cdot F_{n-1} + F_{n-2}$$

* Note: $F_n(1, 1, \ldots, 1) = F_{n-1}(1, \ldots, 1) + F_{n-2}(1, \ldots, 1)$

Fibonacci numbers!

* Used to calculate the growth of rabbit population.

* [BIZ'18] Rabbit approximates!

Narayana's Cows Sequence

[Narayana Pandita, Ganita Kaumudi, 1356]

A cow produces a calf every year. Cows start producing calves from the beginning of the fourth year. Then, starting from 1 cow in the first year, how many cows are there after n years?

$$N_n = N_{n-1} + N_{n-3} \quad \text{with}$$

$$N_0 = N_1 = N_2 = 1.$$

$$N_n := \det \begin{pmatrix} x_1 & 0 & 1 & & & \\ 1 & x_2 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & 1 \\ & & & \ddots & \ddots & 0 \\ & & & & 1 & x_n \end{pmatrix}$$

$$* \quad N_n = x_n \cdot N_{n-1} + N_{n-3}$$

Thm (restated)

$\{N_n\}$ is complete for VF.

Padovan Sequence

[Richard Padovan, 1980s]

$$P_n = P_{n-2} + P_{n-3}$$

$$P_n := \det \begin{pmatrix} 0 & -x_1 & 1 & & & \\ 1 & 0 & -x_2 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & 1 \\ & & & \ddots & \ddots & -x_{n-1} \\ & & & & 1 & 0 \end{pmatrix}$$

$$P_n = x_{n-1} P_{n-2} + P_{n-3}$$

<u>Thm</u>: $\{P_n\}$ is complete for VF.

Proofs

$$\det \begin{pmatrix} x_{11} & x_{12} & 1 & & & \\ 1 & x_{22} & x_{23} & 1 & & \\ & 1 & x_{33} & x_{34} & 1 & \\ & & \ddots & \ddots & \ddots & 1 \\ & & & \ddots & \ddots & x_{n-1,n} \\ & & & & 1 & x_{n,n} \end{pmatrix}$$

$$= \begin{bmatrix} x_{11} & -x_{12} & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{22} & -x_{23} & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdots$$

$$\cdots \begin{bmatrix} x_{n-1,n-1} & -x_{n-1,n} & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{nn} & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A(z) = \begin{bmatrix} z & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$B(z) = \begin{bmatrix} 0 & z & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Addition :

$$A(f+g) = A(f) \cdot A(0) \cdot A(0) \cdot A(g)$$

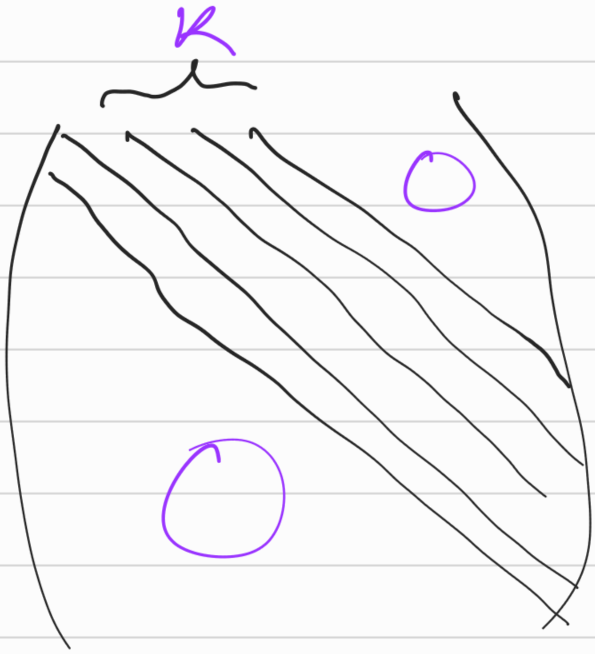
Multiplication : $A(f \cdot g) =$

$$A(0) \cdot A(0) \cdot B(-g) \cdot B(f) \cdot A(0) \cdot B(g) \cdot B(-f)$$

$$B(f) = A(0) \cdot B(-1) \cdot A(0) \cdot A(1) \cdot A(f) \cdot$$
$$A(0) \cdot A(-1) \cdot B(1) \cdot A(0) \cdot A(0)$$

OPEN PROBLEMS

* $(1,k)$ - diagonal matrices



k : bounded $= VF$

$k : \Omega(n^{\delta}) = VBP = VDet$

$k = \Theta(\log n)$?