Points, lines and polynomial identities

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Outline

- Points and lines: Sylvester-Gallai theorem and relatives
- Applications:
  - Locally correctable codes
  - Algebraic identity testing (aka polynomial identity testing)
- Higher degree analog
- Proof sketch

Point-line incidences

Main theme: Given a collection of points and lines satisfying certain properties, bound some combinatorial measure (number of incidences, number of lines, number of points,…)

Many results and conjectures: Szemeredi-Trotter, Guth-Katz (Erdös distinct distance problem), Kakeya,…

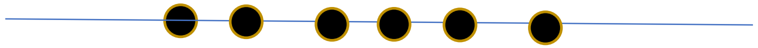
This talk: Sylvester-Gallai theorem and relatives

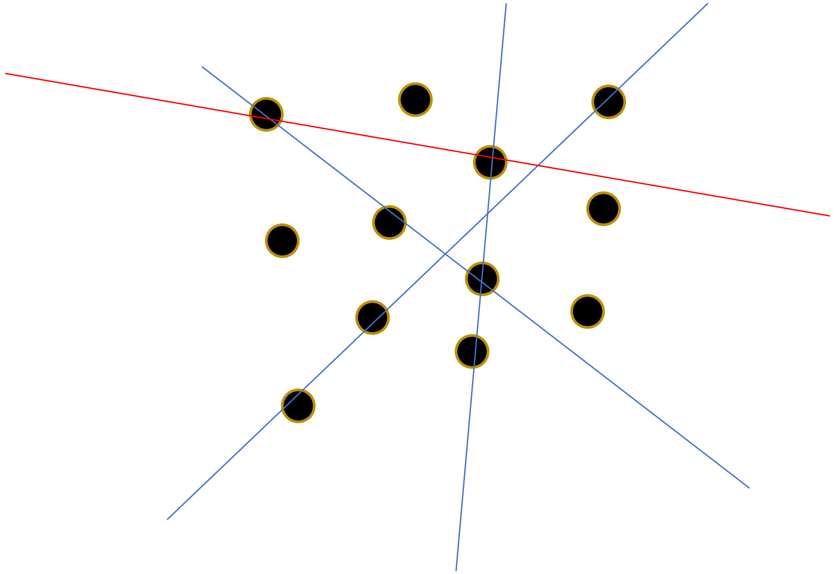
Sylvester-Gallai theorem

Conjectured by Sylvester'93 and Erdös'43, proved by Melchior'41 and Gallai'44:

- A finite set of points $P \subseteq \mathbb{R}^2$
- Any line through any two points in P meets a 3rd point in P (special line)

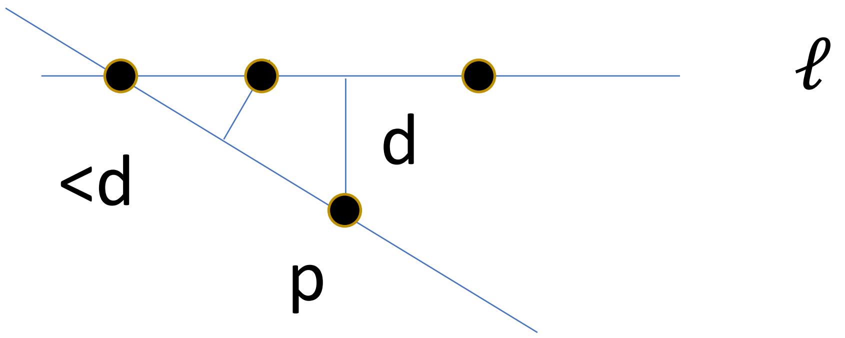
$\Longrightarrow$ Points are colinear (dim(affine-span P)=1)





Proof

Let p and ℓ be the closest point-line pair (line that passes through at least 3 points)



Important: P finite (otherwise P=$\mathbb{R}^2$), over $\mathbb{R}$

Same proof for P$\subseteq\mathbb{R}^n$

Some important relatives

[Kelly'86]: Over $\mathbb{C}$, same condition $\Longrightarrow$ dim(affine-span P) ≤ 2

[Edelstein-Kelly'66]: Colorful version: $P = R \sqcup G \sqcup B$

Every non-monochromatic line contains all 3 colors

$\Longrightarrow$ dim(affine-span P) ≤ 3

[Barak-Dvir-Wigderson-Yehudayoff'11, Dvir-Saraf-Wigderson'12]:
Robust version:

Special lines through every $p \in P$ cover δ-fraction of P

$\Longrightarrow$ dim(affine-span P) $\leq O(1/\delta)$

Algebraic/Dual rephrasing

Finite set of homogeneous linear equations:

$\{L_1(x_1,\dots,x_n),\dots,L_m(x_1,\dots,x_n)\}\subseteq\mathbb{R}[x_1,\dots,x_n]$

Any solution to any two equations also solves a 3rd equation

$\Longrightarrow$ $\dim(\mathrm{span}\{L_i\})\leq 2$ (over $\mathbb{C}$: $\dim(\mathrm{span}\{L_i\})\leq 3$)

Reduction:

Linear equation L: $\langle v,x\rangle=0 \leftrightarrow \mathrm{span}\{v\}$ in $\mathbb{R}^n$

H a hyperplane in general position

point corresponding to L : $p_L = \mathrm{span}\{v\}\cap H$

$L_3 \in \mathrm{span}(L_1,L_2) \Leftrightarrow p_1,p_2,p_3$ colinear

Applications

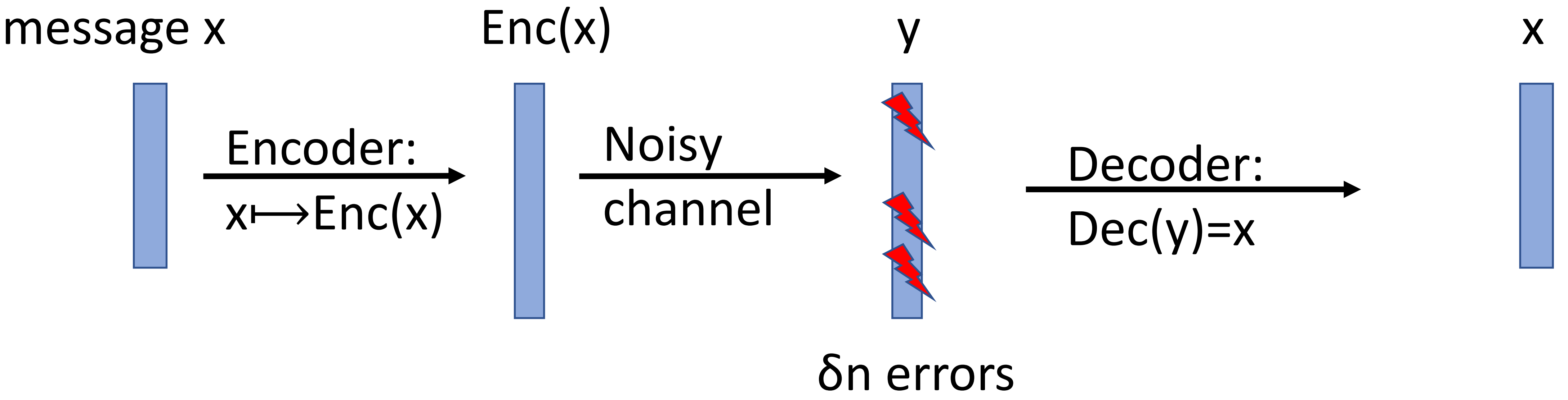
[Dvir-S'05]: SG-type theorem relevant for:

- Locally Correctable Codes (LCCs)
- Polynomial Identity Testing (PIT) of depth-3 circuits

[Beecken-Mittmann-Saxena'13, Gupta'14]:

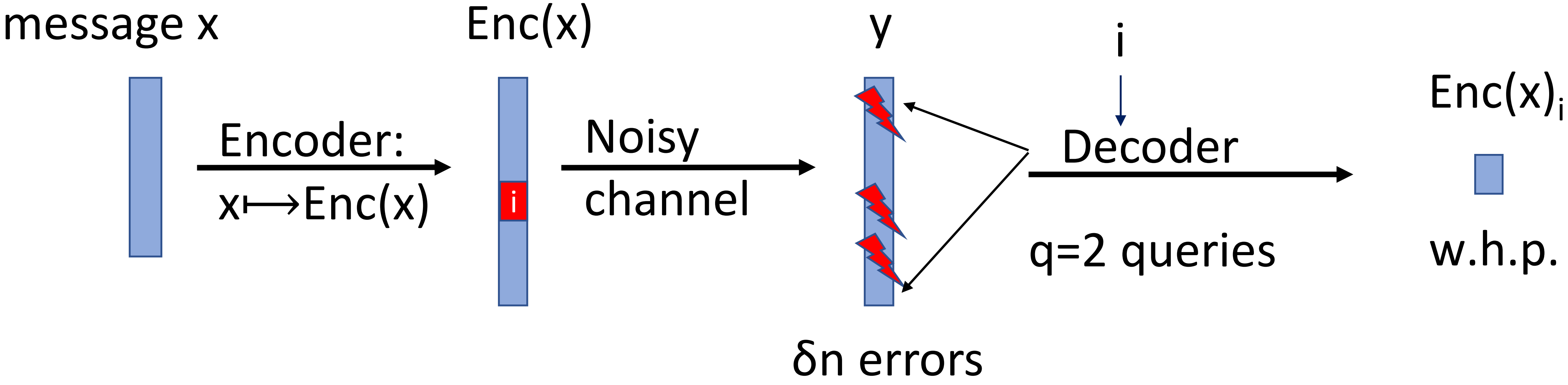
Higher degree version of SG type theorems relevant for PIT of depth-4 circuits

Error correcting codes



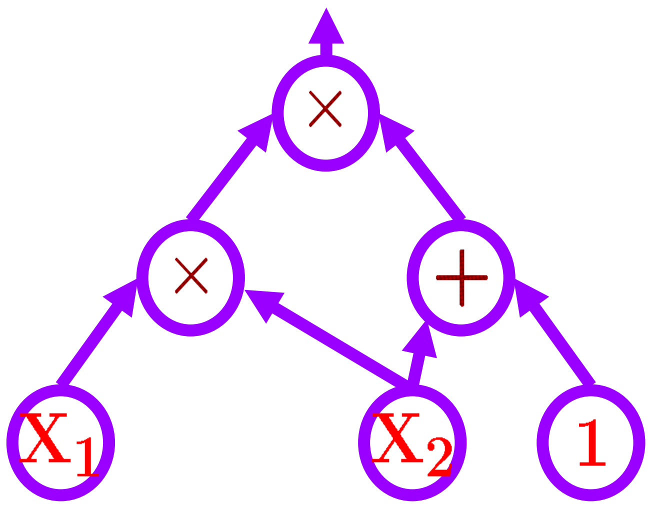
- Many applications in practice (communication, storage) and theory (PCP, crypto,…)
- Typical goals: minimize overhead (i.e. higher rate |x|/|Enc(x)|), decoding from a large fraction of errors (higher δ), efficient decoding

Locally correctable codes



- Locality: super efficient local correction. Is it achievable?
- Assume: Enc is a linear map $Enc(x)_i = L_i(x)$
- If L_i can be recovered from L_j, L_k then they satisfy the SG property
- High probability decoding $\Longrightarrow$ many colinear triplets
- (robust) SG theorem $\Longrightarrow$ Dim(span L_i)=small $\Longrightarrow$ Rate is zero

Polynomial identity testing (PIT)



Model: algebraic circuits (computations using +,×)

Challenge: Given algebraic circuit C decide C(x)=0?

Efficient Randomized algorithm [Schwartz'80, Zippel'79, DeMillo-Lipton'78]

Goal: A proof. I.e., a deterministic algorithm

Motivation:

- Primality testing [Agrawal-Kayal-Saxena'02]
- Parallel algorithms for finding perfect matching [Karp-Upfal-Wigderson'85, Mulmuley-Vazirani-Vazirani'87]
- Efficient deterministic algorithms implies lower bounds [Kabanets-Impagliazzo'03]

Identity testing of depth-3 algebraic circuits

Example: Let $\omega^d=1$ is the following true:

$$\prod_{i=1\ldots d}(3\omega^5X+(2\omega^5-5\omega^i)Y-6\omega^iZ) +$$
$$\prod_{i=1\ldots d}(-2\omega^iX+(3\omega^i+5)Y+(6-5\omega^i)Z) +$$
$$\prod_{i=1\ldots d}((2\omega^{2}-3\omega^i)X-(3\omega^i+2\omega^i)Y+5\omega^2Z) =?\ 0$$

Solution: Let

$U= 3X+2Y$

$V=5X+6Z$

$W=2X-3Y+5Z$

After simple manipulation:

$$\prod(U-\omega^iV) + \prod(V-\omega^iW) + \prod(W-\omega^iU) = (U^d-V^d) + (V^d-W^d) + (W^d-U^d) = 0$$

Identity testing of $\sum\prod\sum$ circuits

Let $A=\prod a_i$, $B=\prod b_i$, $C=\prod c_i$, $a_i,b_i,c_i \in \mathbb{R}[x_1,\ldots,x_n]$ linear forms

Decide whether A+B+C=0

First nontrivial case (A+B=0 verified by unique factorization)

[Dvir-S'05]: If we set $a_i=b_j=0$ then $\exists k$ such that $c_k=0$, can use colorful SG

[Kayal-Saraf'09]: If A+B+…+M=0 then (morally) $\dim(\{a_i\},\{b_i\},\ldots,\{m_i\})=m^{O(m)}$

PIT algorithm: Find basis, expand and verify identity in O(1) variables

[Saxena-Seshadhri'11]: BB-PIT for m summands in $n^{O(m)}$ time (any field)

[Gupta-Kamath-Kayal-Saptharishi'13]: PIT for $\sum\prod\sum$ (unbounded degree)
$\Longrightarrow$ PIT for general circuits

Identity testing of $\sum^{[3]}\prod\sum\prod$ circuits

Let $A=\prod a_i$, $B=\prod b_i$, $C=\prod c_i$, $a_i,b_i,c_i \in \mathbb{R}[x_1,\ldots,x_n]$ degree d polynomials

Decide whether A+B+C=0

Theorem[Agrawal-Vinay'08] : PIT for homogeneous depth-4 $\Longrightarrow$ PIT for general circuits

Conjecture [Beecken-Mittmann-Saxena'13, Gupta'14]:
If A+B+C=0 disjoint then algebraic-rank($\{a_i\},\{b_i\},\{c_i\}$)=O(1)

Intuition: If we set $a_i=b_j=0$ then there is some k such that $c_k=0$.
Need degree d Edelstein-Kelly theorem (colorful degree d SG)

Example: a=xy+zw, b=xy-zw, $c_1\cdot c_2\cdot c_3\cdot c_4$ = (x+z)(x+w)(y+z)(y+w)

Problem: Product vanishes when a=b=0 but not always the same c_k

Our results

Higher degree SG type theorems

$A=\{a_i\}$ quadratic polynomials

- For every a_i,a_j there is a_k that vanishes whenever a_i and a_j do

  [S'19] $\Rightarrow \dim(\{a_i\})=O(1)$

  if $A=R\sqcup G\sqcup B$ … $\Rightarrow \dim(\{a_i\})=O(1)$

- For every a_i,a_j whenever a_i and a_j vanish then so does $\prod_{k\neq i,j} a_k$

  [Peleg-S'20] $\Rightarrow \dim(\{a_i\})=O(1)$

- $A=\prod a_i$, $B=\prod b_i$, $C=\prod c_i$, quadratic polynomials

[Peleg-S'21] If $A+B+C=0$ disjoint (wlog) then $\dim(\{a_i\},\{b_i\},\{c_i\})=O(1)$
(via colorful version of [Peleg-S'20])

Answers [Beecken-Mittmann-Saxena'13, Gupta'14] for degree $d=2$

Proof ingredients

Main tool I: Algebraic Structure Theorem

Theorem[S'19,Peleg-S'20]: $Q_1,Q_2,\{P_i\}$ quadratics s.t. $Q_1(v)=Q_2(v)=0 \Longrightarrow \prod P_i(v)=0$

Then one of the following cases must hold:

1. Some P_i is in the linear span of Q_1, Q_2
2. $\exists$ linear functions ℓ_1,ℓ_2 s.t. $\ell_1\ell_2 \in \text{span}\{Q_1, Q_2\}$
3. $\exists$ linear functions ℓ_1,ℓ_2 s.t. $Q_1 = Q_2 = 0$ modulo ℓ_1,ℓ_2

Examples:

2. $Q_2 = Q_1 + \ell\ell'$, $P_1 = (Q_1 + \ell\ell_1)$ $P_2 = (Q_1 + \ell'\ell_2)$
3. $Q_1 = xa+yb$, $Q_2=xc+yd$, $P_1 = (ad-bc)$, $P_2 = x$, $P_3 = y$

Proof idea: Analyzing how the resultant of Q_1,Q_2 factorizes
Different cases roughly correspond to different degrees of factors

Main tool II: Robust version of E-K theorem

Recall [Edelstein-Kelly'66]: Colorful version: P=R⊔G⊔B

Every non-monochromatic line contains all 3 colors

$\Longrightarrow$ dim(affine-span P) ≤ 3

Robust-EK-Thm [S'19]: P= R⊔G⊔B s.t. every point in one set spans with a δ-fraction of points in the other two sets a point in the third set

$\Longrightarrow$ dim(affine-span P) $= O(1/\delta^3)$

Remark: probably not tight

(rough) Proof outline of [S'19,Peleg-S'20,Peleg-S'21]

Use the algebraic structure theorem to argue that either

- Coefficient vectors of quadratic polynomials satisfy the robust-SG/EK theorem (and we are done), or
- Each quadratic is a function of a few linear functions
- Then show that these linear functions satisfy the conditions of the robust-SG/EK theorem themselves

Intuition: If (vector of coefficients of) a polynomial Q is on many special lines, then Q has a very restricted structure

Actual proofs: A lot of case analysis

Follow up and related work

SG:

- [de Oliveira-Sengupta'22]: SG for cubic polynomials (for every two cubics there exists a third…) by extension of structure theorem to cubics
- [Peleg-S'22,Garg-de Oliveira-Sengupta'22]: Robust Quadratic-SG theorem (for every Q_i, for δ-fraction of Q_j, there exists a Q_k …)

PIT:

- [Limaye-Srinivasan-Tavenas'21]: $n^{n^{\varepsilon}}$ PIT for bounded depth circuits
- [Dutta-Dwivedi-Saxena'21]: Quasi-polynomial time BB PIT for $\sum^{[O(1)]}\prod\sum\prod^{[\log(n)^{O(1)}]}$ using a different techniques

Conclusion

Saw applications of problems in discrete geometry in

- Locally correctable codes
- Verifying algebraic identities

Saw generalization to algebra-geometric questions that are also relevant for identity testing

Many open questions – higher degrees, more sets,…

Thank You!