Subrank of Tensors

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Joint work with Christandl, Derksen, Gesmundo, Makam

- We study a notion in algebraic complexity theory called the subrank of tensors, which measures how much a tensor can be diagonalized

- The subrank was introduced by Strassen in 1987 to study fast matrix multiplication algorithms

- and has connections to several problems in math and physics

- our results:

  1. Random tensors

     We determine the subrank for random tensors

  2. Asymptotic gaps

     We determine gaps in the rate of growth of subrank under powering

- Improve on previous bounds of Strassen & Bürgisser (1987-1991)

1. Subrank and Applications

2. Subrank of random tensors

3. Upper bound

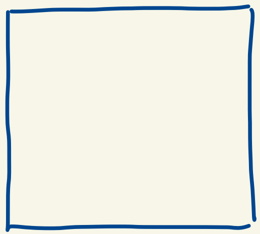
4. Lower bound ingredient: tensor space decomposition

5. Application: non-additivity of subrank

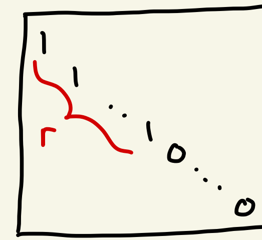
6. Asymptotic gaps

1. Subrank and Applications

Matrix rank

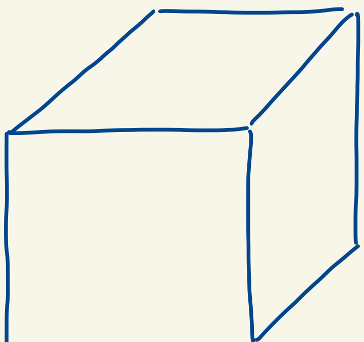
$M =$ 

linear combinations
of rows and columns
⇝

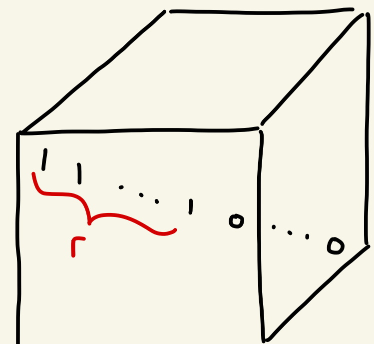


max r

Subrank

$T =$ 

linear combinations
of slices in all
three directions
⇝



max r

Subrank is different from tensor rank!

Tensor rank

minimize

$$T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$$

Equiv:

$$T = U \otimes V \otimes W \cdot \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$$

$R(T)$

Applications

- Matrix multiplication
- Circuit complexity [Raz]

Subrank

maximize

$$\sum_{i=1}^{s} e_i \otimes e_i \otimes e_i = U \otimes V \otimes W \cdot T$$

$Q(T)$

- Matrix Multiplication
- Additive Combinatorics

Applications of Subrank

- **Complexity Theory** — number of independent scalar multiplications that can be reduced to a bilinear map

  used in recursive constructions of matrix mult. algos.

- **Quantum Information** — measure of entanglement (of GHZ type)

- **Combinatorics** — upper bound on hypergraph independence

  E.g. cap sets, sunflowers, corners, ...

Relation to other parameters

$T \in \mathbb{F}^{n \times n \times n}$

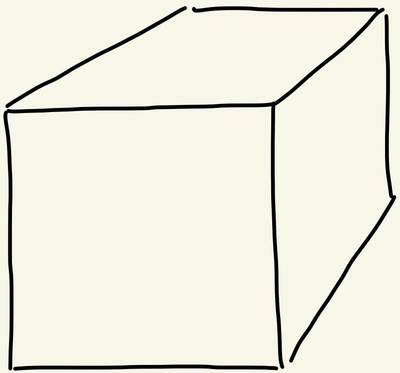
$$0 \le Q(T) \le SR(T) \le n \le R(T) \le n^2$$

$AR(T)$

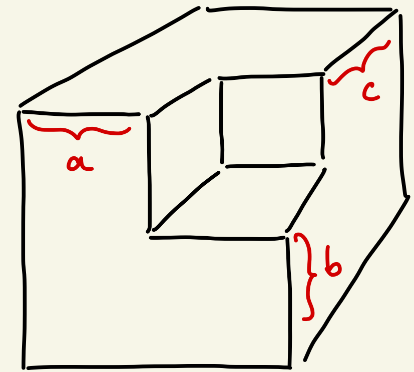
$GR(T)$

$R^G(T)$

Slice rank



linear combinations of slices in all three directions $\rightsquigarrow$



minimize $a+b+c$

$SR(T)$

Geometric rank

$\operatorname{codim} \{ (u,v) \in \mathbb{F}^n \times \mathbb{F}^n : \forall w \;\; T(u,v,w) = 0 \}$

$GR(T)$

$\vdots$

Relation to other parameters

$T \in \mathbb{F}^{n \times n \times n}$

$$\begin{array}{ccccccccc}
0 \le & Q(T) & \le & SR(T) & \le & n & \le & R(T) & \le n^2 \\
 & & & AR(T) & & & & & \\
 & & & GR(T) & & & & & \\
 & & & R^G(T) & & & & & \\
\text{On random tensors } T. & \underbrace{\qquad\quad}_{?} & & \underbrace{\qquad\qquad}_{\approx n} & & & & \underbrace{\qquad\qquad}_{\approx n^2} &
\end{array}$$

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- Application: Subrank is not additive under direct sum.

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<u>Lemma 1</u> $Q(n) =$ largest r such that $\dim C_r = \underbrace{\dim \mathbb{F}^{n\times n\times n}}_{n^3}$.

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Lemma 2 $\dim C_r \le n^3 - r(r^2 - 3n + 2)$

Let $t = Q(n)$

Then $n^3 = \dim C_t \le n^3 - t(t^2 - 3n + 2)$.

Then $t^2 - 3n + 2 \le 0$

So $t \le \sqrt{3n-2}$

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- Non-injective parametrization of C_r
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$X_r = \{ \text{tensors in } \mathbb{F}^{n \times n \times n} \text{ with } [r] \times [r] \times [r] \text{ subtensor arbitrary diag.} \}$

$$\Psi_r : GL_n \times GL_n \times GL_n \times X_r \to \mathbb{F}^{n \times n \times n}$$
$$(A, B, C, T) \longmapsto (A \otimes B \otimes C) T \qquad \text{has image } C_r$$

$\square$

4. Tensor space decompositions

Goal: write tensor space $\mathbb{F}^{n \times n \times n}$ as a sum of tensor subspaces, as efficiently as possible such that each subspace has the form of an $n \times n$ matrix subspace tensored with $\mathbb{F}^n$

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$X[1] = \mathbb{F}^n \otimes X \subseteq \mathbb{F}^{n \times n \times n}$

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Theorem there are subspaces $X_i \subseteq \mathrm{Mat}_{3,3}$ of dim 3 each, such that

$$\mathbb{F}^{3\times 3\times 3} = X_1[1] + X_2[2] + X_3[3].$$

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Theorem There are subspaces $X_i \subseteq (\mathbb{F}^n)^{\otimes n-1}$ of dim n^{n-2} each, such that

$$(\mathbb{F}^n)^{\otimes n} = X_1[1] + X_2[2] + \cdots + X_n[n].$$

Again: dimensions match.

5. Application: Subrank is not additive under direct sum

<u>Theorem</u> There are tensors $S, T \in \mathbb{F}^{n \times n \times n}$ such that $Q(S), Q(T) \leq \sqrt{3n-2}$ while $Q(S \oplus T) \geq n$.

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Proof idea

- Let T be "random".
- Let $S = I_n - T$. Then S is "random".
- Then $Q(S), Q(T) \leq \sqrt{3n-2}$ by our theorem.
- On the other hand, $Q(S \oplus T) \geq Q(S+T) = Q(I_n) = n$. $\square$

6. Asymptotic gap in the subrank

$S \in V_1 \otimes V_2 \otimes V_3$

$T \in W_1 \otimes W_2 \otimes W_3$

Kronecker product: $S \boxtimes T \in (V_1 \otimes W_1) \otimes (V_2 \otimes W_2) \otimes (V_3 \otimes W_3)$

Subrank is super-multiplicative: $Q(S \boxtimes T) \geq Q(S)\,Q(T)$

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Exactly one of the following is true:

i) $T = 0$

ii) $Q(T^{\boxtimes n}) = 1$

iii) $Q(T^{\boxtimes n}) = 1.88^{n - o(n)}$

iv) $Q(T^{\boxtimes n}) \geqslant 2^n$

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Proof idea Classification

i) $T = 0$

ii) T has flattening rank one

iii) T is equivalent to the W-tensor

iv) T restricts to $2 \times 2 \times 2$ diagonal.

Selected Open Problems

1. Our upper bound $Q(T) \leq \lfloor \sqrt{3n-2} \rfloor$ for generic $T \in \mathbb{F}^{n \times n \times n}$ is tight for $n \leq 100$. Is this always true?

2. Determine all possible tensor space decompositions

3. What is the largest gap between $Q(S \oplus T)$ and $Q(S) + Q(T)$?

4. What are the next asymptotic gaps?