Exercise sheet 4

CS242 Formal specification and verification - Autumn 2007

A **partial order** is a binary relation $\leq$ over a set P which is reflexive, antisymmetric, and transitive. That is to say: for all a, b, and c in P:

$a \leq a$ (reflexivity);
if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry);
if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).

A set with a partial order is called a **partially ordered set** (also called a **poset**).

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1. Which of the following sets is a poset under the binary relation β?

The positive integers with m β n if m divides n without leaving any remainder.
The set of computer science students ordered by age, so that X β Y if X was born not later than Y was born.
The set of students on campus with X β Y if X and Y live in the same hall of residence.
The set of churches in Coventry, where C β D if D is within a quarter of a mile radius of C.
The set of words in the dictionary where v β w if w does not come after v.
The set of sets of letters that can be used to make a four letter word in the dictionary (possibly using the same letter more than once), ordered by S β T if S is a subset of T.
The set of positive integers with m β n if m is greater than n.

What role do assumptions about the nature of the sets specified play in determining your answers?

2. Find an example of a binary relation r on a set P to satisfy each of the following conditions:

(a) r is reflexive and antisymmetric but not transitive
(b) r is reflexive and transitive but not antisymmetric
(c) r is antisymmetric and transitive but not reflexive

A **lattice** is a set L with two binary operations, $\vee$ and $\wedge$ such that the following identities hold for all elements a, b, and c of L:

Associative laws:	$a \vee (b \vee c) = (a \vee b) \vee c$	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$
Idempotent laws:	$a \vee a = a$	$a \wedge a = a$
Commutative laws:	$a \vee b = b \vee a$	$a \wedge b = b \wedge a$
Absorption laws:	$a \vee (a \wedge b) = a$	$a \wedge (a \vee b) = a$

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3. Given that L is a lattice prove that the relation $\leq$ defined by $a \leq b$ if $a \wedge b = a$ is a partial order..

4. Suppose that P is partially ordered by $\leq$ in such a way that for any pair of elements x and y in P:

$\exists$t such that $x \leq t$ and $y \leq t$, and that if z is any element such that $x \leq z$ and $y \leq z$, then $t \leq z$.
$\exists$b such that $b \leq x$ and $b \leq y$, and that if z is any element such that $z \leq x$ and $z \leq y$, then $z \leq b$.

In this context, t and z are uniquely defined elements respectively known as the least upper bound (denoted by $x \vee y$) and the greatest lower bound (denoted by $x \wedge y$) of x and y. Verify that P is then a lattice with the binary operations $\wedge$ and $\vee$.

5. Wikipedia omits the idempotent laws from the basic laws for a lattice, and cites them as "important identities" that can be deduced from the other laws. Can you justify this claim?

6. A lattice is distributive if it is such that the following identities hold for all elements a, b, and c of L:

Distributive laws: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ $\quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Draw Hasse diagrams to represent each of these posets:

- the subsets of {1,2,3} ordered by inclusion.
- the divisors of 12 ordered by $x \leq y$ if x divides y exactly.
- the semantically distinct formula that can be derived from the propositional atoms p, q and r using the logical connectives $\wedge$ and $\vee$, ordered by $e \leq f$ if $e \rightarrow f$ is valid.

Show that each poset defines a distributive lattice. How would you interpret the operations $\wedge$ and $\vee$ in each case?

7. Identify which of the six diagrams A, B, C, D, E and F on the attached sheet:

(a) is the Hasse diagram of a poset?
(b) is the poset associated with a lattice?
(c) is the poset associated with a distributive lattice?

Justify your answers.

8. NAND is a binary operation on propositional formulae defined by

$$p \text{ NAND } q = \neg(p \wedge q)$$

Prove that all 16 propositional logic formula in the propositional atoms p and q can be expressed using the operator NAND alone.

Which of the following laws apply to NAND as a binary operator: the associative laws? the commutative laws? the idempotent laws? Justify your answer.

Can you explain why two formula expressed using p, q and NAND alone are semantically equivalent if and only if they evaluate to the same set when p and q are replaced by the sets {1,2} and {1,3}, and X NAND Y is interpreted as "the complement of $X \cap Y$ in the set {1,2,3,4}"?
