Proving the completeness of natural deduction

What is to be proved?

To establish completeness, we seek to prove that, if

$\Phi_1, \Phi_2, \ldots, \Phi_n \models \Psi$

then $\Phi_1, \Phi_2, \ldots, \Phi_n \vdash \Psi$ is a valid sequent..

The premise here is that any assignment of truth values to propositional atoms that makes all the formulae $\Phi_1, \Phi_2, \ldots, \Phi_n$ true also makes the formula Ψ true. The conclusion we wish to draw is that we can use natural deduction to derive Ψ from $\Phi_1, \Phi_2, \ldots, \Phi_n$.

We can reformulate the sequent as stating the universal truth, independent of what truth values are assigned to propositional atoms, of a formula, viz:

$\models \Phi_1 \rightarrow (\Phi_2 \rightarrow \ldots, (\Phi_n \rightarrow \Psi) \ldots))$

Our objective is then to demonstrate the validity of the sequent:

$\vdash \Phi_1 \rightarrow (\Phi_2 \rightarrow \ldots, (\Phi_n \rightarrow \Psi) \ldots))$.

In other terminology, we aim to prove that every *tautology* is a *theorem*.

Proposition

Let Φ be a formula such that $p_1, p_2, \ldots, p_n$ are its only propositional atoms. Let r be any row of the truth table of Φ and let $q_1, q_2, \ldots, q_n$ be defined in such a way that, for i from 1 to n:

$q_i = p_i$ if the entry for p_i in row r is T;

$q_i = \neg p_i$ if the entry for p_i in row r is F.

Then

$q_1, q_2, \ldots, q_n \vdash \Phi$ is provable if the entry for Φ in row r is T

$q_1, q_2, \ldots, q_n \vdash \neg\Phi$ is provable if the entry for Φ in row r is F.

Sketch proof of proposition

The proof proceeds by structural induction on the formula Φ. Informally, ths means that we shall assume the proposition is valid for formulae with fewer logical connectives than Φ. By applying the induction hypothesis, we shall be able to assume that we can use natural deduction to derive each of these subformulae or its negation from the premises $q_1, q_2, \ldots, q_n$. We shall then be able to provide the mini-proofs required to extend these proofs to establish the validity of one or other of the sequents - as appropriate - that is specified in the Proposition.

The decomposition of Φ into structurally simpler subformulae is accomplished by looking at what logical connective appears at the root of the parse tree. There are two possibilities to consider: the connective at the root is $\neg$, or it is one of the binary connectives $\rightarrow$, $\wedge$ or $\vee$.

The connective at the root is ¬:

In this case, we can write Φ as $\neg\Phi_1$.

If Φ evaluates as T on the row r of the truth table, then Φ_1 evaluates as F. Since Φ_1 is a smaller formula than Φ (having one fewer logical connectives), the inductive hypothesis applies to it, and it follows that the sequent:

$$q_1, q_2, ..., q_n \vdash \neg\Phi_1$$

is provable. But this establishes the provability of $q_1, q_2, ..., q_n \vdash \Phi$ since Φ *is* $\neg\Phi_1$.

If Φ evaluates as F on the row r of the truth table, then Φ_1 evaluates as T. Since Φ_1 is a smaller formula than Φ (having one fewer logical connectives), the inductive hypothesis applies to it, and it follows that the sequent:

$$q_1, q_2, ..., q_n \vdash \Phi_1$$

is provable. By invoking the $\neg\neg$ introduction rule, this gives a proof of the validity of the sequent:

$$q_1, q_2, ..., q_n \vdash \neg\neg\Phi_1$$

This establishes the provability of $q_1, q_2, ..., q_n \vdash \neg\Phi$ since Φ is $\neg\Phi_1$.

The connective at the root is one of the binary connectives

There are three possibilities to consider:

- The connective is $\rightarrow$
- The connective is $\wedge$
- The connective is $\vee$

The nature of the argument will be illustrated with one of these connectives, and the completion of the proof left as an exercise to the reader.

Suppose that Φ is $\Phi_1 \rightarrow \Phi_2$.

If Φ evaluates as F on the row r of the truth table, then $\Phi_1 \rightarrow \Phi_2$ evaluates as F. This necessarily means that Φ_1 evaluates as T and Φ_2 evaluates as F. Since Φ_1 and Φ_2 are smaller formulae than Φ (having one fewer logical connectives), the inductive hypothesis applies to them, and it follows that the sequents:

$$q_1, q_2, ..., q_n \vdash \Phi_1$$

$$q_1, q_2, ..., q_n \vdash \neg\Phi_2$$

is provable. To show that the sequent $q_1, q_2, ..., q_n \vdash \neg\Phi$ is provable, it remains to apply the rules of natural deduction to show that the sequent:

$$\Phi_1, \neg\Phi_2 \vdash \neg(\Phi_1 \rightarrow \Phi_2)$$

is valid. This is easily accomplished by proof-by-contradiction, since the assumption that $\Phi_1 \rightarrow \Phi_2$ combined with the premise Φ_1 generates the conclusion Φ_2.

If Φ evaluates as T on the row r of the truth table, then $\Phi_1 \rightarrow \Phi_2$ evaluates as T. This can be interpreted as saying that either Φ_1 evaluates as F or Φ_2 evaluates as T. We shall show how the argument proceeds in the case that Φ_1 evaluates as F, and leave the other possibility to the reader.

If Φ_1 evaluates as F, then the sequent:

$$q_1, q_2, ..., q_n \vdash \neg\Phi_1$$

is provable by the inductive hypothesis, and this extends to a proof of the validity of the sequent $q_1, q_2, ..., q_n \vdash \Phi$ subject to using the rules of natural deduction (Ex. to reader) to show the validity of the sequent:

$\neg\Phi_1 \vdash \Phi_1 \rightarrow \Phi_2$

The above argument deals with the case when the logical conective at the root is →. The same essential principles are used to deal with the other connectives. (Note that the argument in H&R is a little more long-winded, since it distinguishes the four possible ways in which truth values can be assigned to Φ_1 and Φ_2 even though in general this is unnecessary - for instance, the case when Φ_1 evaluates as F in the argument applied to → covers either possible truth value being assigned to Φ_2.)

Having proved the proposition, it is apparent that - in the case of a tautology - every sequent of the form:

$q_1, q_2, ..., q_n \vdash \Phi$

is valid no matter what row of the truth table is considered - i.e. no matter what assignment to the propositional atoms $p_1, p_2, ..., p_n$ is made. To complete the completeness proof, it is enough in effect to observe that the disjunction of all the possible truth assignments to the propositional atoms $p_1, p_2, ..., p_n$ is a tautology. By the law of the excluded middle, exactly one of these assignments of truth values pertains in any partcular model, but this is sufficient to establish the validity of the sequent $\vdash \Phi$ since every sequent of the form

$q_1, q_2, ..., q_n \vdash \Phi$

is valid. This is a sort of generalisation of the binary version of the ∨ elimination rule that perhaps requires a more formal justification. It can be provided by applying the law of the excluded middle n times in order to partitiion the possible assigments on truth values to $p_1, p_2, ..., p_n$ into 2^n singleton classes. This partition is pictorially represented in H&R in the figure on p53.
