Propositional logic: Natural deduction

CS242 Formal Specification and Verification

University of Warwick

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Natural deduction

Proving sequents of the form

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

using proof rules.

John and Jane arguments symbolically:

$$p \land \neg q \rightarrow r, \neg r, p \vdash q$$

 $p \land \neg q \rightarrow r, \neg r, \neg p \vdash q$

Rules for conjunction

Introduction:

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i$$

Elimination:

$$\frac{\phi \wedge \psi}{\phi} \wedge e_1 \qquad \frac{\phi \wedge \psi}{\psi} \wedge e_2$$

$$\frac{\phi \wedge \psi}{\psi} \wedge e_2$$

Example proof:

$$p \land q, r \vdash q \land r$$

Exercises 1.2:

1.(b)
$$p \land q \vdash q \land p$$

1.(c)
$$(p \land q) \land r \vdash p \land (q \land r)$$

Rules of double negation

$$\frac{\neg \neg \phi}{\phi} \neg \neg e \qquad \frac{\phi}{\neg \neg \phi} \neg \neg i$$

Example proof:

$$p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$$

Rules for implication

Modus ponens:

$$\frac{\phi \qquad \phi \rightarrow \psi}{\psi} \rightarrow e$$

Modus tollens:

$$\frac{\phi \to \psi \quad \neg \psi}{\neg \phi} \text{MT}$$

Example proofs:

$$p, p \rightarrow q, p \rightarrow (q \rightarrow r) \vdash r$$
 $p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q$

Motivation for formal proofs

- Proofs are fundamental in mathematics.
- In propositional logic, ϕ is provable if and only if ϕ is valid, and the problem of checking this is decidable. However,
 - this problem is NP-complete;
 - equivalence of provability and validity, and their decidability, do not hold for other important logics.
- Some applications of formal proofs:
 - formal methods;
 - artificial intelligence.

Implies-introduction:

$$\begin{bmatrix}
\phi \\
\vdots \\
\psi
\end{bmatrix}$$

$$\phi \to \psi$$

Example proofs:

$$eg q
ightarrow
eg p
ightarrow p
ightarrow p
ightarrow (
eg p
ightarrow p
ightarrow$$

Formulas ϕ such that $\vdash \phi$ are called *theorems*. If

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

is provable, then so is

$$\vdash \phi_1 \to (\phi_2 \to (\cdots \to (\phi_n \to \psi) \cdots))$$

General points about boxes

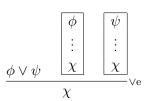
- ► The premises and the conclusion of any complete proof must be outside boxes.
- Any box must begin with its assumptions, and end with its conclusion. These must be outside any inner box.
- ▶ Boxes can be nested. Before closing a box, all boxes opened within that box must be closed. In a complete proof, all boxes must be closed.
- ▶ When applying a proof rule, its premises must be earlier in the proof, and not within boxes which have been closed.

Rules for disjunction

Introduction:

$$\frac{\phi}{\phi \vee \psi} \vee i_1 \qquad \qquad \frac{\psi}{\phi \vee \psi} \vee i_1$$

Elimination:



Example proofs:

$$p \lor q \vdash q \lor p$$
$$p \land (q \lor r) \dashv \vdash (p \land q) \lor (p \land r)$$

The copy rule

$$\frac{\phi}{\phi}$$
copy

Example proof:

$$\vdash p \rightarrow (q \rightarrow p)$$

Exercises 1.2:

1.(I)
$$p \rightarrow q, r \rightarrow s \vdash (p \lor r) \rightarrow (q \lor s)$$

1.(n)
$$(p \lor (q \rightarrow p)) \land q \vdash p$$

$$1.(\mathsf{q}) \, \vdash q \to (p \to (p \to (q \to p)))$$

Rules for negation

Bottom-elimination:

$$\frac{\perp}{\phi}$$
 \(\perp \)

Not-elimination:

$$\frac{\phi - \neg \phi}{\bot} \neg e$$

Not-introduction:

$$\begin{array}{c}
\phi \\
\vdots \\
\bot \\
\neg \phi
\end{array}$$

Example proofs:

$$\neg p \lor q \vdash p \to q$$

$$p \to q, p \to \neg q \vdash \neg p$$

$$p \to \neg p \vdash \neg p$$

$$p \land \neg q \to r, \neg r, p \vdash q$$

Derived rules

Modus tollens.

Not-not-introduction.

Reductio ad absurdum:



Tertium non datur (law of excluded middle):

$$\frac{}{\phi \vee \neg \phi}$$
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Example proof:

$$p \rightarrow q \vdash \neg p \lor q$$