Propositional logic: Natural deduction

CS242 Formal Specification and Verification

University of Warwick

Autumn term 2006

Natural deduction

Proving sequents of the form

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

using *proof rules*.

John and Jane arguments symbolically:

$$\begin{aligned} p \wedge \neg q \rightarrow r, \neg r, p &\vdash q \\ p \wedge \neg q \rightarrow r, \neg r, \neg p &\vdash q \end{aligned}$$

Rules for conjunction

Introduction:

$$\frac{\phi \qquad \psi}{\phi \wedge \psi} \wedge \mathrm{i}$$

Elimination:

$$\frac{\phi \wedge \psi}{\phi} \wedge \mathrm{e}_1 \qquad \frac{\phi \wedge \psi}{\psi} \wedge \mathrm{e}_2$$

Example proof:

$$p \wedge q, r \vdash q \wedge r$$

Exercises 1.2:

1.(b) $p \wedge q \vdash q \wedge p$

1.(c) $(p \wedge q) \wedge r \vdash p \wedge (q \wedge r)$

Rules of double negation

$$\frac{\neg\neg\phi}{\phi}\,\neg\neg\mathrm{e} \qquad \frac{\phi}{\neg\neg\phi}\,\neg\neg\mathrm{i}$$

Example proof:

$$p, \neg\neg(q \land r) \vdash \neg\neg p \land r$$

Rules for implication

Modus ponens:

$$\frac{\phi \qquad \phi \to \psi}{\psi} \to\mathrm{e}$$

Modus tollens:

$$\frac{\phi \to \psi \qquad \neg\psi}{\neg\phi} \mathrm{MT}$$

Example proofs:

$$p, p \to q, p \to (q \to r) \vdash r$$

$$p \to (q \to r), p, \neg r \vdash \neg q$$

Motivation for formal proofs

- Proofs are fundamental in mathematics.
- In propositional logic, ϕ is provable if and only if ϕ is valid, and the problem of checking this is decidable. However,
  - this problem is NP-complete;
  - equivalence of provability and validity, and their decidability, do not hold for other important logics.
- Some applications of formal proofs:
  - formal methods;
  - artificial intelligence.

Implies-introduction:

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \to \psi} \to\mathrm{i}$$

Example proofs:

$$\neg q \to \neg p \vdash p \to \neg\neg q$$

$$\vdash p \to p$$

$$\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$$

$$p \wedge q \to r \dashv\vdash p \to (q \to r)$$

Formulas ϕ such that $\vdash \phi$ are called *theorems*.
If

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

is provable, then so is

$$\vdash \phi_1 \to (\phi_2 \to (\cdots \to (\phi_n \to \psi) \cdots))$$

General points about boxes

- The premises and the conclusion of any complete proof must be outside boxes.
- Any box must begin with its assumptions, and end with its conclusion. These must be outside any inner box.
- Boxes can be nested. Before closing a box, all boxes opened within that box must be closed. In a complete proof, all boxes must be closed.
- When applying a proof rule, its premises must be earlier in the proof, and not within boxes which have been closed.

Rules for disjunction

Introduction:

$$\frac{\phi}{\phi \vee \psi}\vee\mathrm{i}_1 \qquad\qquad \frac{\psi}{\phi \vee \psi}\vee\mathrm{i}_2$$

Elimination:

$$\frac{\phi \vee \psi \qquad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\vee\mathrm{e}$$

Example proofs:

$$p \vee q \vdash q \vee p$$

$$p \wedge (q \vee r) \dashv\vdash (p \wedge q) \vee (p \wedge r)$$

The copy rule

$$\frac{\phi}{\phi}\text{copy}$$

Example proof:

$$\vdash p \rightarrow (q \rightarrow p)$$

Exercises 1.2:

1.(l) $p \rightarrow q, r \rightarrow s \vdash (p \lor r) \rightarrow (q \lor s)$

1.(n) $(p \lor (q \rightarrow p)) \land q \vdash p$

1.(q) $\vdash q \rightarrow (p \rightarrow (p \rightarrow (q \rightarrow p)))$

Rules for negation

Bottom-elimination:

$$\frac{\bot}{\phi}\,\bot\mathrm{e}$$

Not-elimination:

$$\frac{\phi \qquad \neg\phi}{\bot}\,\neg\mathrm{e}$$

Not-introduction:

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\,\neg\mathrm{i}$$

Example proofs:

$$\neg p \vee q \vdash p \to q$$

$$p \to q, p \to \neg q \vdash \neg p$$

$$p \to \neg p \vdash \neg p$$

$$p \wedge \neg q \to r, \neg r, p \vdash q$$

Derived rules

Modus tollens.
Not-not-introduction.
Reductio ad absurdum:

$$\dfrac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\,\mathrm{RAA}$$

Tertium non datur (law of excluded middle):

$$\dfrac{}{\phi \vee \neg\phi}\,\mathrm{LEM}$$

Example proof:

$$p \rightarrow q \vdash \neg p \vee q$$